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A CONTINUOUS TIME STORAGE MODEL WITH MARKOV NET INPUTS

MAJ NELSON S. PACHECO

MAY 1980

FINAL REPORT

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A model for a dam is considered wherein the net input rate (input minus output rate) follows a finite Markov chain in continuous time, X(·), and the dam contents process, C(·), is the integral of the Markov chain. The dam is then modelled with the bivariate Markov process (X(·),C(·)), of which three variations are considered. These are the doubly-infinite dam with no top or bottom, the semi-infinite dam with only one boundary, and the finite dam with both a top and a bottom. Some of the analysis is performed under the most general situation in

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For the semi-infinite and finite dams, techniques of invariance used in the physical sciences are introduced to study first passage times. These techniques are used to derive the distribution and moments of the wet period of the dam in special cases, and the limiting probabilities of emptiness and overflow.

A CONTINUOUS TIME STORAGE MODEL WITH MARKOV NET INPUTS

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CHAPTER I

INTRODUCTION

1.1 Preliminaries

The field of storage theory has been of long-standing interest in the engineering community, and has in the last few decades become one of the most energetically pursued fields in applied probability. Although the origin of storage theory lies in the now classical study of sizing of water reservoirs by hydrologists, the recent developments have transcended these original applications. The interest of probabilists and statisticians in the interesting mathematical and statistical problems which have arisen from a study of storage problems have in fact led to a new branch in hydrology known as stochastic hydrology. Many of the techniques developed by the stochastic hydrologists have in turn become of independent interest, so that at present this theory has developed on its own merits as a mathematical construct, rather than being strictly tied to the classical study of reservoir sizing. Many natural linkages have also been established between storage theory and other areas in stochastic processes, among them queuing theory and the theory of stationary processes.

As a result of the extensiveness of the field, it is necessary for contemporary investigators to narrow substantially the field of inquiry to specific types of models and specific aspects of those models. This investigation studies certain problems associated with a continuous-time storage model which has a Markovian structure. Before specifically addressing the problems studied in this

investigation, however, we will give a, necessarily brief, overview of the history and categorizations of storage theory.

1.2 History and Literature Review

The classic object of storage theory is the study of a water reservoir, or dam, which is fed by some source of water such as a river. The size of the dam to be constructed is, of course, dependent on the amount of water which can be expected from the supply source. The size of the dam should be such that the probability of occurrence of either overflow or emptiness is minimized.

The classical analysis of this problem specified certain deterministic functions for both the amount of water input and the amount of water drawn off, and from this the optimal size of the reservoir was determined. This type of analysis dates at least as far back as 1883, when it was treated by W. Rippl.

The stochastic nature of the problem was first addressed by A. Hazen (1914) and later by C. E. Sudler (1927). These two individuals considered the data representing river runoffs as having an associated uncertainty. Although their methods of analysis were crude by today's standards, it is both to their credit and an indication of the interesting nature of the problems in storage theory that they developed the techniques of using "probability paper" and data simulation, respectively, in their studies.

H. E. Hurst [26]* in 1951 studied the problem of determining the reservoir storage required on a given stream to guarantee a given draft by considering the cumulative sums of the departure of the

*Numbers in brackets refer to bibliography.

annual totals from the mean annual total discharge. These cumulative departures can be thought of as the contents of a hypothetical dam with no top or bottom; or the unrestricted contents, measured from some point representing the mean contents.

Under this model, the level of the dam in year n, C_n, is the result of n net yearly inputs (input minus output) $X_1, X_2, \ldots X_n$. Thus we have $C_n = \sum_{i=1}^{n} X_i$, so that the contents become the well-studied partial sums process. The functional studied by Hurst for this model is the range up to time n, R_n, given by

$$R_n = \max_{1 \leq i \leq n} C_i - \min_{1 \leq i \leq n} C_i .$$

Hurst and subsequent investigators used R_n for initial sizing estimates by concluding that the range up to time n gives an indication of the size of a dam which would have been required to contain that amount of water without either overflowing or becoming empty. Hurst also considered certain other variations on the range as defined above, such as an adjusted range obtained by subtracting off the average up to time n. The basic idea for the sizing of the reservoir, however, is the same.

If the interval size in this discrete-time model is sufficiently large (yearly, for example), then it may be reasonable to suppose that the summands X_i are stochastically independent. This was the first model studied, and in this case the unrestricted contents process becomes a random walk.

The significance of the Hurst paper lies in the extremely long records of data which he compiled, some extending up to 2000 years, as was the case for the records on the Nile river. From these records he

drew a conclusion on the rate of growth of the range which contradicted the random walk model described above. For a random walk model it is well known that the expected value of the range grows as $n^{0.5}$. However, Hurst found in his analysis of the data that all of these quantities showed a surprisingly similar exponent of 0.69 to 0.80 with a mean of .72. This is indeed much too large to be explained by this model and has been the object of much subsequent study.

This anomaly, now known as the Hurst phenomenon, drew the attention of W. Feller [21] who in 1951 derived the asymptotic distribution and moments of the range for the iid case by appealing to approximations by Brownian motion. Feller mentioned that the Hurst phenomenon might be explained by assuming that the summands are not independent, although as noted later by P. Moran [33], the dependence would have to be strange indeed, for any reasonable model will in fact have an asymptotic growth of 0.5. Moran then commented that the Hurst behavior may be pre-asymptotic in nature, and that the records which Hurst studied were not long enough to reach their asymptotic values.

Subsequently Moran in 1964 [34] obtained the mean range when the inputs were iid but with a symmetric stable distribution with parameter γ and found that the mean range varied as $n^{1/\gamma}$.

This, then, represents the two main theories advanced to the present time to explain the Hurst phenomenon; non-iid pre-asymptotic behavior, and iid heavy-tailed net inputs as represented by the stable inputs.

The heavy tails explanation is not as appealing to many applied hydrologists because of conceptual difficulties involved with an input

which has an infinite variance. However, it is the opinion of this investigator that the pre-asymptotic theory is burdened by the fact that none of the Hurst data seemed asymptotic to $n^{0.5}$. In fact, all of the series which he examined showed remarkably consistent large growth rates for as far back as he could find data.

Although we will not concern ourselves in great detail with the Hurst behavior, we will demonstrate that the continuous-time model which we investigate does in fact exhibit the Hurst behavior pre-asymptotically for moderate values of time.

The more modern studies in storage theory began with Moran in a series of papers from 1954 - 1957 in which he studied dams with a variety of inputs and operational policies. In his analysis, Moran considered the finite dam with both a top and bottom and performed the analysis in both discrete time and continuous time.

In the continuous time approach, Moran assumed that the input process was an additive homogeneous process; that is, a process with stationary independent increments. This means that the input increments, say $C(t_2) - C(t_1)$ in non-overlapping intervals (t_1, t_2) are independent and have a distribution which depends only on $t_2 - t_1$. This is a straight analog of the situation in discrete time when the successive inputs are considered to be independent. Although Moran's continuous time analysis is a more convenient model for a real dam in that one is no longer limited to studying the inputs only at certain time epochs, it nevertheless has the fault that as the time intervals become shorter, the assumption of independence in successive intervals becomes less realistic. Among other contributors to the model is

J. Gani [23], who considered the input process to be Poisson with a unit release rate.

E. Lloyd [30] in 1963 was the first to consider a dam, in discrete time, where the successive inputs were dependent. He supposed that the input X_n in successive time intervals $(n, n+1)$ followed a Markov chain and the release rate in each interval was a constant, say r. The dam contents at time n are then given by

$$C_n = \min(a, \max(0, C_{n-1} + X_n - r))$$

where a is the capacity of the dam and $0 \leq X_0 \leq a$.

This model, which has since become known as the Lloyd dam, received considerable attention in the discrete time case in both this version and the semi-infinite topless version where $a = \infty$. Notable contributions to this study include Ali Khan and Gani [1], who studied the time to first emptiness for the semi-infinite dam, and Ali Khan [2], who considered the finite case. In all of these studies the Markov chain has a finite state space. Brockwell and Gani [11] considered the time to first emptiness for the case in which the Markov chain has the non-negative integers as state space.

As far as range analysis for the unrestricted contents with dependence is concerned, F. Gomide [24] treated the case of Markovian inputs in discrete time, and B. Troutman [41] studied limiting distributions for the discrete time process with Markovian and certain stationary inputs using a weak convergence approach.

If the continuous time model of a storage system is a better approximation than the corresponding discrete time model, and the Lloyd model is a better approximation than the independent input model, then a continuous time version of the Lloyd dam offers a much closer

correspondence between theory and a realistic storage system. Relatively few studies have been done on this model, which is the subject of this investigation. Most notable of the published results have been those by McNeil [32] and Brockwell [12]. McNeil studied a dam in continuous time in which the input process follows a two-state Markov chain with one of the states being zero. When the input process is in the zero state, the dam is being drained, and he supposes that there is a general measurable function of the dam level, say $g(x)$, which represents the demand rate. Although he sets up the problem in this generality, he is able to obtain explicit results only for the cases when $g(x)$ is constant and $g(x)$ is exponential. McNeil is able to derive the limiting distribution of the contents and first passage times.

Brockwell considers the case in which the net input rate follows a general Markov chain with a finite state space, and by setting up Kolmogorov-type equations is able to derive the limiting distribution of the contents and first times to emptiness and overflow.

1.3 Objectives and General Approach

The model considered in this investigation is the one formulated by Brockwell where the net input, say X_t, is a Markov chain in continuous time defined on a finite state space $\{\mu_1,\ldots,\mu_m\}$. In analogy with discrete-time models, the dam content at time t, C_t, is given by the integral of the net input process up to time t. Hence (for a doubly infinite dam)

$$C_t = \int_0^t X_u du, \qquad t \geq 0 .$$

This formulation is the continous-time analog of the unrestricted contents process described earlier. To contrast it with the next model which we consider, we will refer to the bivariate process (X_t, C_t) as the doubly-infinite dam. When we refer to the marginal process C_t, the term "unrestricted contents" will also be used.

If we now refine this model by assuming that the contents must be non-negative then

$$C_t^T = \int_0^t X_u^* du, \quad t \geq 0 \quad \text{where}$$

$$X_u^* = \begin{cases} 0 & \text{if } C_u^T = 0 \text{ and } X_u < 0 \\ X_u & \text{otherwise} \end{cases}$$

and we will refer to the bivariate process (X_t, C_t^T) in this context as the semi-infinite dam. The terminology topless dam has also been used for models of this type. A symmetric variation of this is to restrict the top but not the bottom, so that, if a is the highest level which the contents can attain, then

$$C_t^B = \int_0^t X_u^* du, \quad t \geq 0 \text{, where}$$

$$X_u^* = \begin{cases} 0 & \text{if } C_u^B = a \text{ and } X_u > 0 \\ X_u & \text{otherwise.} \end{cases}$$

We will refer to this variation of the semi-infinite dam as the bottomless dam.

The third model which we consider is one for which the contents must be non-negative and can not exceed a certain level, say a. In this case the contents are given by

$$C_t^F = \int_0^t X_u^* du, \quad t \geq 0, \quad \text{where}$$

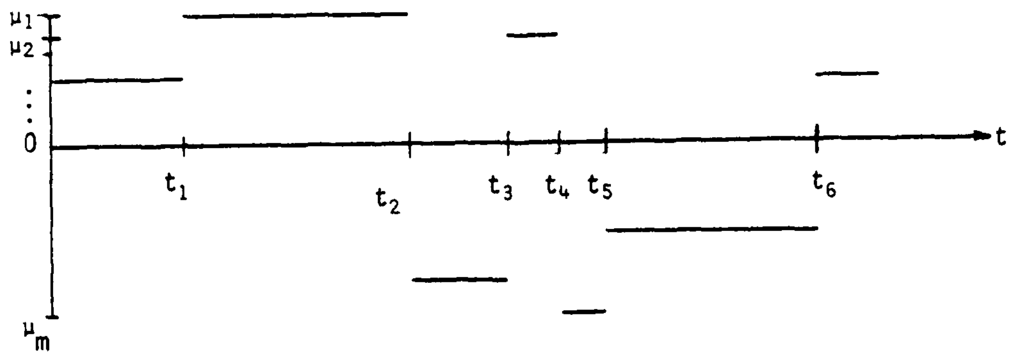
$$X_u^* = \begin{cases} X_u & \text{if } C_u^F \in [0,a) \text{ and } X_u > 0 \\ X_u & \text{if } C_u^F \in (0,a] \text{ and } X_u < 0 \\ 0 & \text{otherwise.} \end{cases}$$

In this case we will refer to the bivariate process (X_t, C_t^F) as the finite dam.

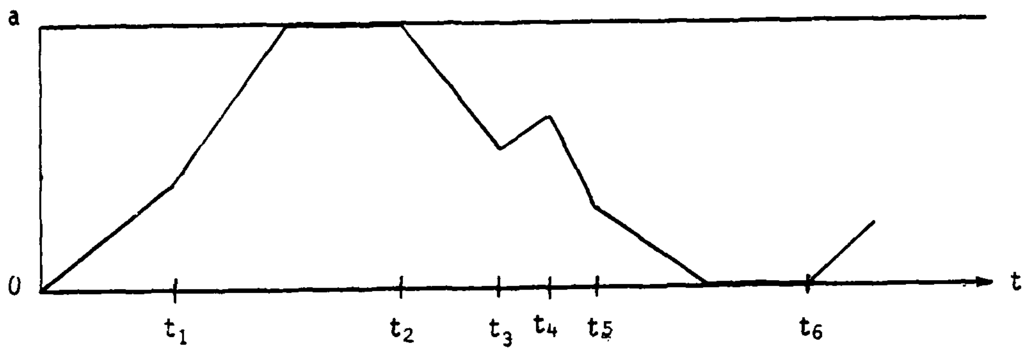
Now, in simple terms a Markov process is one in which the distribution of the future, given the present, is independent of the past. The C_t process does not meet the requirement and is not, in fact, Markov. This is due to the fact that the present only conveys information on the level of the dam contents, and not on the rate at which the contents are changing. Obviously, adding this information will affect the future distribution of the contents. The bivariate process (X_t, C_t) , however, does contain the information on the rate of change of the contents and is a bivariate Markov process. The state space of such a process is discontinuous in nature, since it evolves on a set of disjoint lines as shown in Figure 1.1 for the finite dam. The solid dots at the right boundaries for the positive rates and the left boundaries for the negative rates represent the fact that when the process hits these points it remains there for a random time. A solid dot on the right boundary, for example, represents an overflow condition while a solid dot on the left boundary represents an emptiness condition. Markov processes with discontinuous state spaces of this type were studied by Moyal [35].

In Chapter II we present a brief review of background material which is prerequisite to the development of later chapters. Although none of the material is new, the results which we quote here are scattered through various sources and so we include it here in a unified manner as an aid to the reader.

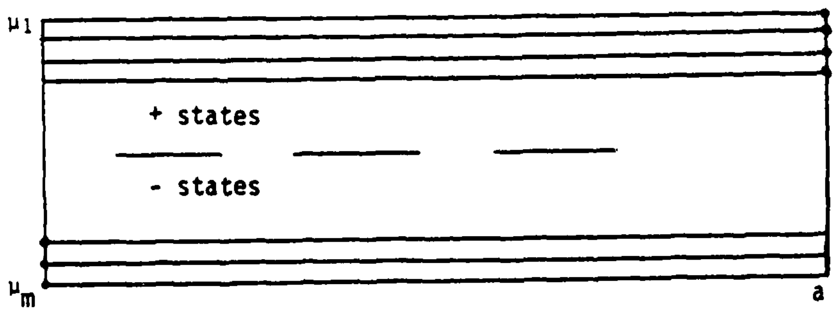
In Chapter III we study the first and second moment functions of the marginal processes for the doubly infinite dam. These functions are useful for fitting the model and as an aid in the initial estimates of parameters. In this chapter we prove a generalization of a result



a. Sample paths of X_t process.



b. Sample path of C_t process for the finite dam.



c. State space of (X_t, C_t) process for the finite dam.

Figure 1.1 Sample paths and state space of (X_t, C_t).

reported by McNeil [32] that the autocorrelation function for the marginal contents process does not depend on the states of the Markov chain in the two-state case.

In Chapter IV we begin with the analysis of the range for the doubly infinite dam. The relevance of this to dam sizing studies has been discussed earlier. Although the range variable has been studied extensively in discrete time, this is the first analysis in continuous time. We first study the joint distribution of the maximum and minimum variables by finding its Laplace transform. We do this for the most general situations in which the Markov chain is defined on m states with an arbitrary generator. We call this the <u>General Case</u>. An inversion of this transform would provide the joint distribution from which the distribution of the range could, in principle, be obtained. The inverse, however, is not obtainable in a simple analytic form and although a numerical procedure could be used, we choose to restrict ourselves to an analytic rather than numerical analysis of this problem. By restricting ourselves to special cases, we are able to obtain exact expressions for the expected range for all t, and an asymptotic distribution for the range. The special cases which we consider are those in which the Markov chain is defined on two states, which we call the <u>Two-State Case</u> and the subcase for which the two states μ_1 and μ_2, and the two holding time parameters, λ and ρ, have the relation $\mu_1 = -\mu_2$ and $\lambda = \rho$. We call this the <u>Symmetric Case</u>.

We are able to derive an explicit solution for the expected range function, ER_t, in the symmetric case. We do this by first obtaining the marginal distributions of the maximum and minimum for the two-state

case. The expected range can be expressed, using the linearity of expectation, as the difference in expected value of the maximum and minimum. We obtain the expected range function in its Laplace transformed version. Fortunately, the Laplace transform can be expanded in a neighborhood of infinity and the resulting series can be inverted term-by-term as a confluent hypergeometric series. The properties of the confluent hypergeometric series have been thoroughly studied (see, e.g., Buchholz [15]). Tabled values are available in Jahnke, Emde, and Lösch [27].

For illustrative purposes, we chose a particular subcase and performed a precise numerical calculation of ER_t which we include as a graph for $0 \leq t \leq 10$ in Figure 4.1. For numerical calculation purposes, Kummer's first formula (Buchholz) was quite useful, for it allowed calculation in positive terms rather than in terms of a slowly converging alternating series.

Examination of the graph of ER_t shows remarkably close behavior to that reported by Hurst for moderate values of t. Therefore we include in this chapter a short note on the Hurst phenomenon. Another aspect which is evident from the graph is the extremely rapid convergence of ER_t to its asymptotic value of $\sqrt{\frac{8}{\pi}}\ t^{1/2}$. This indicates that for moderate values of t, as small as 5, very good approximations can be obtained from asymptotic results. In conclusion, we discuss the order of convergence by performing an asymptotic expansion of the Laplace transform of ER_t in a neighborhood of zero, as discussed in Doetsch [19].

In Chapter V we discuss what we call, in general, invariance methods as applied to the analysis of the semi-infinite and the finite

dams. The invariance techniques to which we refer are those developed by V. A. Ambarzumian [3] and Chandrasekhar [16] in the 1940's for the study of radiative transfer in stellar atmospheres, and subsequently refined by Bellman *et al.* [4-7] in the late 1950's, and early 1960's in the field of Neutron Transport Theory. Invariance techniques are very closely related to the idea of regeneration in probability theory. In this chapter we present a detailed discussion of the principle of invariance as applicable to our model.

The principle of invariance presented the astrophysicists with a powerful tool for the solution of certain physical problems which had been at best laboriously solved using more classical techniques. The same held true in the applications to Neutron Transport Theory. In studying the problems of first passage times in the storage model considered herein, a strong relation between all of these problems becomes evident. This is particularly true in the study of the wet period, or the sojourn time of the process from the time that it leaves the zero state until the time that it returns to the zero state. The classical formulation for the distribution of this variable involves setting up Kolmogorov equations, which lead to a two-point boundary value problem in the finite dam. The invariance techniques, however, enable us to solve directly for the transform of this distribution, leading to a system of algebraic equations in the case of the semi-infinite dam, and to an initial value problem in the case of the finite dam. In the symmetric case, we are able to invert the transform and thus obtain the distribution of the wet period. More generally, we are able to solve for the expected value of the wet period in the two-state case, which in turn leads us to finding

necessary and sufficient conditions for recurrence for the semi-infinite dam.

We finish this chapter by using a renewal argument to calculate the limiting probability of emptiness for the topless and finite dams, and the limiting probability of overflow for the bottomless dam.

In Chapter VI we establish a Functional Central Limit Theorem for the doubly infinite dam on two states. We prove that the process $C(nt)/\sqrt{n}$ converges weakly in the zero drift case to the Wiener process as n goes to infinity, that is, $C(n\cdot)/\sqrt{n} \Rightarrow W(\beta\cdot)$ on $D[0,\infty)$ where β is a function of the four parameters. We therefore have the approximation $P[C(n\cdot)/\sqrt{n} \in B] \simeq P[W(\beta\cdot) \in B]$, for $B \in \mathcal{B}$, the Borel field induced by the open sets relative to the $D[0,\infty)$ metric. This is a substantial improvement on the results of Fukushima and Hitsuda [22] and Pinsky [38], who were only able to show convergence of the marginal distributions.

We establish weak convergence by the method of establishing an embedded partial sum process. The difficulty here is that a direct appeal to Donsker's Theorem is not possible because the sum contains a renewal counting function as an upper index. Fortunately, we are able to overcome this difficulty by a technique similar to that used by Resnick and Durrett [20] who consider weak convergence of sums with random indices.

As an application of the weak convergence which we establish, we use the continuous mapping theorem to establish the asymptotic distribution of the range, since the corresponding result for the Wiener process was established by Feller [21].

Chapter VII presents a summary of the results and recommendations for the further study of these problems.

CHAPTER II

BACKGROUND MATERIAL

2.1 Markov Processes

In this chapter we present a very brief summary of definitions and results which will be needed in subsequent chapters. Proofs are not provided for most theorems, since they can be found in any text containing Markov processes, such as Çinlar [18] or Chung [17].

Def. 2.1: The real-valued stochastic process $\{X_t, t \geq 0\}$ is a continuous-time Markov Process iff $\forall n$ and $\forall t_1 < t_2 < \ldots < t_n < t_{n+1}$,

$$P\left[X_{t_{n+1}} \varepsilon B | X_{t_1}, \ldots, X_{t_n}\right] = P\left[X_{t_{n+1}} \varepsilon B | X_{t_n}\right] \text{a.s. for}$$

$B \varepsilon \mathcal{B}(R)$, the linear Borel sets.

This is often stated as the future, conditioned on the present, being independent of the past. We remark that in the above definition X_t may be a vector, which is in fact the case that we will consider later.

Def. 2.2: A function p from $R_+ \times R \times \mathcal{B}(R)$ into $[0,1]$ is called a transition probability function (tpf) iff

(1) $p(t,x,\cdot)$ is a probability measure on $\mathcal{B}(R)$ $\forall t,x$

(2) For each $B \varepsilon \mathcal{B}(R)$, $p(.,.,B)$ is product measurable with respect to $\mathcal{B}(R_+) \times \mathcal{B}(R)$

(3) p satisfies the Chapman-Kolmogorov relation:

$\forall B \varepsilon \mathcal{B}(R)$, $s,t\ R_+$,

$$p(t+s,x,B) = \int_{y \varepsilon R} p(s,x,dy)p(t,y,B)$$

A tpf gives the probability of transitioning from some starting position to some Borel set in some fixed time.

Def. 2.3: We define the Markov Process to have stationary transition probabilities $P(\cdot,\cdot,\cdot)$ iff

$$P[X_{t+s} \in B \mid X_t] = p(s, X_t, B) \quad \text{a.s.}$$

To construct a Markov Process with stationary transition probabilities all that is needed is the tpf and an initial distribution measure $\pi(\cdot)$ on $\mathcal{B}(R)$ as follows from the following theorem.

Theorem 2.1: Given a transition function p, define the following finite-dimensional distribution functions for $0 = t_0 < \ldots < t_n$:

$$F_{t_0,\ldots,t_n}(B_0 x \ldots x B_n)$$

$$= \int_{B_0 x \ldots x B_n} \cdots \int \pi(dx_0) p(t_1 - t_0, x_0, dx_1) p(t_2 - t_1, x_1, dx_2)$$

$$\ldots p(t_n - t_{n-1}, x_{n-1}, dx_n)$$

where π is some initial probability measure on $\mathcal{B}(R)$. Then $\{F_{t_0,\ldots,t_n}\}$ is a consistent family and hence by the Kolmogorov Consistency Theorem we are guaranteed a process $\{X_t\}$ on $\Omega = R^{[0,\infty)}$. $\{X_t\}$ has stationary transition probabilities and is a Markov process.

For convenience, we will label P_π the measure constructed on $(R^{[0,\infty)}, \mathcal{B}(R^{[0,\infty)}))$ from π and $p(t,x,B)$. When $\pi(B) = \delta_x(B)$ then we write $P_x = P_\pi$.

2.2 Strong Markov Processes

Suppose the Markov process is defined on a background probability space $(\Omega,\mathcal{B},P)$. Suppose that on this probability space we have an increasing continuum of Borel fields $\mathcal{B}_t$, $t \geq 0$, so that $\mathcal{B}_t \subset \mathcal{B}_s$ if $s > t$.

Def. 2.4: A random variable $T:\Omega \to [0,\infty]$ is a stopping time with respect to $\{\mathcal{B}_t\}$ iff $(T \leq t) \in \mathcal{B}_t \forall t$. The pre-T sigma field is defined by: $\mathcal{B}_T = \{\Lambda \in \mathcal{B} \mid \Lambda(T \leq t) \in \mathcal{B}_t \forall t\}$.

Intuitively, $\mathcal{B}_T$ encompasses all of the information obtained by observing the process up until time T. It can be easily checked that $\mathcal{B}_T$ is indeed a Borel field. The previous definitions have been independent of the Markov process. We now suppose that $\{\mathcal{B}_t\}$ is the set of Borel fields generated by the Markov process, i.e.,

$$\mathcal{B}_t = \mathcal{B}(X_s,\ s \leq t), \text{ where } \mathcal{B}(X_s,\ s \leq t) \text{ is}$$

the smallest Borel field generated by the random variables $\{X_s,\ s \leq t\}$. With this in mind, we will define a strong Markov process:

Def. 2.5: A Markov process $\{X_t\}$ is said to be strong Markov if $\forall$ stopping times T and $\forall\, x \in R$, $\forall\, A \in \mathcal{B}\,(R^{[0,\infty]})$, $P_x[X(\cdot+T) \in A \mid \mathcal{B}_T] = P_{X(T)}[x(\cdot) \in A]$ as P_x.

Thus we see that a strong Markov process is one for which the Markov property holds for stopping times.

2.3 Finite State Markov Chains

When the state space is a subset J of the integers, we call the process a Markov Chain and we can package the tpf into a convenient matrix form, known as the transition matrix, defined as follows:

Def. 2.6: A function $\underset{\sim}{P}(t) = (P_{ij}(t))_{i,j\varepsilon J}$ is called a transition matrix if

(i) $P_{ij}(t) \geq 0$, $\sum_j P_{ij}(t) \leq 1 \quad \forall\, t \geq 0$

(ii) $P_{ij}(t)$ is measurable

(iii) The Chapman-Kolmogorov relation holds:

$$\underset{\sim}{P}(t+s) = \underset{\sim}{P}(t)\, \underset{\sim}{P}(s)$$

If in (i) we have $\sum_j P_{ij}(t) < 1$ we call $\underset{\sim}{P}$ substochastic. If, moreover,

(iv) $\underset{\sim}{P}(t) \to I$ as $t \searrow 0$ we call $\underset{\sim}{P}$ standard.

The subsequent discussion will only involve finite state, standard, stochastic transition matrices.

We have seen that given an initial distribution and a transition matrix, the Markov chain is probabilistically determined in the sense that all of the finite dimensional distributions are determined. Since this is a determining class (Billingsley [9]), then two different Markov chains with the same initial distributions and transition matrices are probabilistically indistinguishable.

From the statement of our storage problem, however, it will be more natural to specify the infinitesimal transition probabilities, that is, $P_{ij}(\delta t)$ for $\delta t \simeq 0$. Specification of these infinitesimal

probabilities lead, in the finite state space case, to a direct solution for the transition matrix which, along with the initial distribution, will specify the Markov chain.

Def. 2.7: The waiting time random variable W_t is defined by

$W_t = \inf \{s > 0 | X_{t+s} \neq X_t\}$, and $W_t = \infty$ if the set is empty.

W_t is the time spent in state X_t before jumping to a different state.

Def. 2.8: The jump times $\{T_i\}_{i=0}^{\infty}$ are defined by

$T_0 = 0$

$T_1 = W_0$

$T_{n+1} = T_n + W_{T_n}$

Def. 2.9: The sequence of states visited, $\{X_n\}_{n=0}^{\infty}$ are defined as

$X_n = X(T_n)$

$\{X_n\}$ is also known as the embedded jump chain.

We now state some well-known results.

Theorem 2.2: The Markov property implies that $\forall i \varepsilon J \; \exists \; \lambda_i \varepsilon [0,\infty) \ni$

$P[W_t > u | X_t = i] = e^{-\lambda_i u}$, $u \geq 0$

Hence the holding times are exponential with a parameter dependent on the state. The following holds for joint distributions:

Theorem 2.3: $\forall n$, $j \varepsilon J$, $u \geq 0$

$$P\,[X_{n+1} = j,\ T_{n+1} - T_n > u \mid X_0,\ldots,X_n,T_0,\ldots,T_n] = \pi_{X_n,j}\, e^{-\lambda_n u}$$

where $\underset{\sim}{\pi}$ is a stochastic matrix $\ni \pi_{ij} \geq 0$, $\pi_{ii} = 0$, $\underset{\sim}{\pi}\,\underset{\sim}{1} = \underset{\sim}{1}$. By taking conditional expectations of both sides of the equation with respect to X_0, ..., X_n we can show the following:

<u>Theorem 2.4</u>: $P[T_j - T_{j-1} > u_j,\ j=1,\ldots,n \mid X_0,\ldots,X_n] = \prod_{j=1}^{n} e^{-\lambda_{X_{j-1}} u_j}$

Hence the times between jumps $\{T_j - T_{j-1}\}_{j \leq n}$ are conditionally independent and exponentially distributed.

As a corollary of this theorem, note that if the sequence of states visited is deterministic, then the times between jumps are unconditionally independent.

By conditioning on the time of the first jump, we obtain the <u>Kolmogorov Backward Equation</u>:

<u>Theorem 2.5</u>: $\forall i,j,t \geq 0$

$$P_{ij}(t) = \delta_{ij}\, e^{-\lambda_i t} + \int_0^t \lambda_i e^{-\lambda_i s} \sum_{\substack{k \in J \\ k \neq j}} \pi_{ik} P_{kj}(t-s)\,ds$$

By conditioning on the time of the last jump before t, we obtain the <u>Kolmogorov Forward Equation</u>:

<u>Theorem 2.6</u>: $\forall i,j,\ t \geq 0$

$$P_{ij}(t) = \delta_{ij} e^{-\lambda_i t} + \int_0^t \sum_{\substack{k \in J \\ k \neq j}} P_{ik}(s) \lambda_k\, ds\, \pi_{kj} e^{-\lambda_j (t-s)}$$

Although the forward equation can lead to difficulties with explosive processes, that is, processes for which $P[X_t \in J] < 1$ for some t, finite state Markov chains are non-explosive and if the jump matrix is irreducible both the forward and backward equation possess the same unique solution.

Theorem 2.7: $\forall i,j, P_{ij}(t)$ has a continuous derivative and

$$P'(0) = Q$$

$$P'(t) = QP(t)$$

where $Q_{ij} = \begin{cases} -\lambda_i, & i=j \\ \lambda_i \pi_{ij}, & i \neq j \end{cases}$

from which,

$$P(t) = e^{Qt} = \sum_{i=0}^{\infty} \frac{Q^i t^i}{i!} \quad .$$

Def. 2.10: The matrix Q is called the generator of the Markov chain. The parameters λ_i^{-1} give the mean holding times in the i^{th} state and π_{ij} gives the probability of a jump from state i to state j in the jump chain.

We note that any matrix Q such that $Q\,\underset{\sim}{1} = \underset{\sim}{0}$, $q_{ij} \geq 0$, $i \neq j$, and $q_i < 0$ is a generator for a Markov chain.

In the subsequent analysis, we will proceed by starting with a generator for the Markov chain net input process X_t. This generator will be specified by the infinitesimal transition probabilities. We will then establish results for the contents process which is derived from the net input process. We will maintain as much generality as possible throughout. The most general setting possible is that on an arbitrary number of states with a generator as described above. We will refer to this as the General Case. In many instances this is much too general to obtain explicit results, and so we restrict ourselves to a Markov chain on two states, which we call the Two-State Case.

In this case we take as a generator

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \rho & -\rho \end{pmatrix} .$$

As we discussed earlier, the generator by itself does not specify the Markov chain; we also need the initial distribution $\underset{\sim}{\pi}$. In many cases we will assume that $\underset{\sim}{\pi}$ is the stationary distribution, which we discuss next.

2.3.1 Stationary Distribution

From the Markov property it follows that if

$$\pi_i(t) = P[X_t = i] \quad \text{and}$$

$$\underset{\sim}{\pi} = (\pi_i)_{i=1}^{n}$$

then

$$\underset{\sim}{\pi}'(t) = \underset{\sim}{\pi}'(0)P(t) .$$

Now, suppose that there is a time-independent solution, say $\underset{\sim}{\pi}$, to

$$\underset{\sim}{x}' = \underset{\sim}{x}' P(t)$$

then if $\underset{\sim}{\pi}(0) = \underset{\sim}{\pi}$, it is clear that $\underset{\sim}{\pi}(t) = \underset{\sim}{\pi} \; \forall \; t \geq 0$.

In this case the Markov chain is in probabilistic equilibrium which we call (strict) stationarity, and $\underset{\sim}{\pi}$ is known as the stationary distribution.

The condition for existence of a solution of the system above can be related to the generator by the following:

Theorem 2.8: $\underset{\sim}{x}'P(t) = \underset{\sim}{x}' \forall t$ iff $\underset{\sim}{x}'Q = 0$. If X_n is irreducible, the solution is unique.

Hence the stationary distribution is the normalized left eigenvector corresponding to the zero eigenvalue, whose right eigenvector is $\underset{\sim}{1}$.

Now, suppose that $\lim_{t\to\infty} P_{ij}(t) = \pi_j < \infty$, independent of i. Then, in matrix notation,

$$\lim_{t\to\infty} P(t) = \underset{\sim}{1}\pi'$$

and $\lim_{t\to\infty} \underset{\sim}{\pi}'(t) = \underset{\sim}{\pi}'(0) \lim_{t\to\infty} P(t) = \underset{\sim}{\pi}'(0)\, \underset{\sim}{1}\underset{\sim}{\pi}' = \underset{\sim}{\pi}'$.

We call $\underset{\sim}{\pi}$ the <u>limiting distribution</u> of the chain, and it is free of the initial distribution. We will show in the next section that for our case the limiting distribution is identical with the stationary distribution.

Of interest is the rate of convergence to the limiting distribution for the net input process. We discuss this next.

2.3.2 <u>Rate of Convergence to the Limiting Distribution</u>

The computation of limiting distributions and results on the rate of convergence are most easily handled by algebraic methods. An excellent reference for algebraic methods in Markov chains is Karlin [28]. The basic tool used is the spectral decomposition theorem for matrices.

<u>Theorem 2.9</u>: If Q is an nxn matrix with distinct eigenvalues θ_i, $i=1,\ldots,n$ and right eigenvectors $\underset{\sim}{t}_i$, $i=1,\ldots,n$ then Q admits the spectral decomposition

$$Q = T\Lambda T^{-1}$$

where $T = (t_1,\ldots,t_n)$ and $\Lambda = \text{diag}\,(\theta_i)$.

Now, the time-dependent transition matrix was given by

$$P(t) = e^{Qt} = \sum_{i=0}^{\infty} \frac{Q^i t^i}{i!}$$

Substituting $Q = T\Lambda T^{-1}$ into the above, we obtain, assuming distinct θ_i's,

$$e^{Qt} = I + \sum_{i=1}^{\infty} \frac{T\Lambda^i T^{-1} t^i}{i!} = I + T(e^{\Lambda t} - I)T^{-1}$$

$$= Te^{\Lambda t}T^{-1}$$

where $e^{\Lambda t} = \text{diag}\left(e^{\theta_i t}\right)$.

Hence

$$P(t) = Te^{\Lambda t}T^{-1} .$$

Now, writing $T^{-1} = \begin{pmatrix} \underset{\sim}{r}_1' \\ \vdots \\ \underset{\sim}{r}_n' \end{pmatrix}$

we can write the above as

$$P(t) = \sum_{i=1}^{n} \underset{\sim}{t}_i \, \underset{\sim}{r}_i' \, e^{\theta_i t}$$

and, since the stationary distribution, $\underset{\sim}{\pi}$, is the left eigenvector of Q corresponding to the zero eigenvalue and right eigenvector $\underset{\sim}{1}$, we have

$$P(t) = \underset{\sim}{1}\,\underset{\sim}{\pi}' + \sum_{i=2}^{n} t_i \, r_i' \, e^{\theta_i t} .$$

We now prove the following lemma concerning the nonzero eigenvalues of a generator.

Lemma 2.1: The non-zero eigenvalues of a generator have negative real part.

Proof: If θ is an eigenvalue then θ satisfies $Q\underset{\sim}{x} = \theta\underset{\sim}{x}$ for some $\underset{\sim}{x} \neq 0$. This says that if $Q_i = \sum_{i \neq j} q_{ij}$,

$$\sum_{j=1}^{n} q_{ij}x_j = \theta x_i \ , \ i=1,\ldots, n$$

or $$-Q_i x_i + \sum_{j \neq i} q_{ij}x_j = \theta x_i$$

so that $$\sum_{j \neq i} q_{ij}x_j = (\theta + Q_i)x_i$$

Now, let $x_k = \bigvee_{1 \le j \le n} |x_j|$

Then, for $i=k$ we get

$$(\theta+Q_k) = \sum_{j\neq k} q_{kj}\left(\frac{x_j}{x_k}\right) \text{ with } \left|\frac{x_j}{x_k}\right| \leq 1$$

So then $|\theta + Q_k| \leq \sum_{j\neq k} |q_{kj}| \left|\frac{x_j}{x_k}\right| \leq \sum_{j\neq k} |q_{kj}| = Q_k$

Hence $|\theta + Q_k| \leq Q_k$, or $|\theta - (-Q_k)| \leq Q_k$

This says geometrically that θ lies within a circle centered at $-Q_k$ with radius Q_k on the complex plane, so that $Re(\theta) < 0$.

With the help of the lemma we see that

$$P(t) = \underset{\sim}{1}\underset{\sim}{\pi}' + \sum_{i=2}^{m} t_i r_i' e^{\theta_i t} \xrightarrow{t\to\infty} \underset{\sim}{1}\underset{\sim}{\pi}',$$

with the rate of convergence being exponential and governed by the largest non-zero eigenvalue.

Now, if $\underset{\sim}{\pi}'(o) = \underset{\sim}{\pi}'$, then

$$\underset{\sim}{\pi}'(t) = \underset{\sim}{\pi}'P(t) = \underset{\sim}{\pi}'e^{Qt} = \underset{\sim}{\pi}' \sum_{i=0}^{\infty} \frac{Q^i t^i}{i!}$$

$$= \underset{\sim}{\pi}' I + \sum_{i=1}^{\infty} \underset{\sim}{\pi}'QQ^{i-1}\frac{t^i}{i!} = \underset{\sim}{\pi}'I + \underset{\sim}{0} = \underset{\sim}{\pi}'$$

since $\underset{\sim}{\pi}' Q = \underset{\sim}{0}$

So we see that the limiting distribution $\underset{\sim}{\pi}$ satisfies $\underset{\sim}{x}'P(t) = \underset{\sim}{x}'$ and, by uniqueness of solutions in the irreducible case, coincides with the stationary distribution.

2.4 Weak Convergence

We present here a brief discussion of the theory of weak convergence which will be used subsequently. For a thorough treatment of the subject, the book by Billingsley [9] is recommended.

We begin with a complete separable metric space (S,ρ) and we suppose that $\mathcal{S}$ is the Borel σ-field generated by the open sets under ρ. The concept of weak convergence involves convergence of sequences of measures defined on the metric space as follows:

Def. 2.11: Suppose $\{P_n\}_{n=0}^{\infty}$ are probability measures defined on (S,ρ).

We say that $\{P_n\}$ converges weakly to P_0, written $P_n \Rightarrow P_0$ iff

$$\int f d\, P_n \to \int f d\, P_0$$

for all bounded continuous real valued functions on S.

Weak convergence of stochastic processes is defined by considering the weak convergence of the induced measures. To do this, it is of benefit to consider random elements.

Def. 2.12: X is a random element of $(S,\mathcal{S})$ if $\exists$ a probability space $(\Omega,\mathcal{B},P)$ such that X is a measurable map from $(\Omega,\mathcal{B},P)$ into $(S,\mathcal{S})$.

With this definition in mind, we can define weak convergence of random elements by

Def. 2.13: If $\{X_n\}$ is a sequence of random elements on $(S,\mathcal{S})$, then X_n converges weakly to X_0 iff $P \circ X_n^{-1} \Rightarrow P \circ X_0^{-1}$, where $P \circ X_n^{-1}$ represents the probability measure on $(S,\mathcal{S})$ induced by X_n.

This type of framework serves for any dimensionality. For example, if $(S,\rho) = (R, |x-y|)$, then a random element is a random variable in

one dimension. If $(S,\rho) = (R^k, [\sum_1^k (x_i-y_i)^2]^{\frac{1}{2}})$, then a random element is a random vector. For our analysis of weak convergence of the contents process, we will work in the space $S = D[0,\infty)$, the space of right-continuous functions on $[0,\infty)$ with finite left limits. Before discussing the $D[0,\infty)$ metric, we will discuss the simpler space $D[0,1]$. Under the uniform convergence metric,

$$\rho^*(x,y) = \sup_{0\le t\le 1} |x(t)-y(t)|$$

the space $D[0,1]$ is not separable. A metric under which $D[0,1]$ is separable is the Skorohod metric,

$$\rho(x,y) = \inf_{\lambda\in\Lambda} \rho^*(\lambda,e) \vee \rho^*(x,y\circ\lambda)$$

where $\Lambda = \{\lambda:[0,1] \to [0,1] \mid \lambda(0) = 0 \;,\; \lambda(1) = 1,\ \lambda$ is continuous, one to one, onto, and strictly increasing $\}$ and e is the identity map. Hence Λ comprises the set of time transformation of $[0,1]$. The idea behind the Skorohod metric is to make functions which are 'close' after a sufficiently small time transformation also close in the metric. Unfortunately, $D[0,1]$ is not complete under the Skorohod metric. However, Billingsley has modified the Skorohod metric into an equivalent metric under which $D[0,1]$ is complete. For details see Billingsley, pg. 112-113. Clearly, the same development holds on $D[0,k]$ for any k .

For processes on $D[0,\infty)$, we desire to define a metric so that random elements X_n will converge weakly, $X_n \Rightarrow X_0$ in $D[0,\infty)$ iff $X_n \Rightarrow X_0$ in $D[0,k]$ for any k .

To accomplish this, let Λ_k be the set of homeomorphisms from $[0,k]$ onto $[0,k]$ with the properties described earlier for Λ .

For $x,y \varepsilon D[0,k]$, define the Skorohod metric

$$\tilde{d}_k(x,y) = \inf_{\lambda \varepsilon \Lambda_k} \rho^*_k(\lambda,e) \vee \rho^*_k(x,y\cdot\lambda)$$

where ρ^*_k is the uniform metric on $[0,k]$.

Let d_k be the Billingsley modification to $\tilde{d}_k$ which makes $(D[0,k],d_k)$ a complete separable metric space. Now let $D^\infty = \prod_{k=1}^{\infty} D_k$, so that if $x \varepsilon D^\infty$ then $x = (x_1, x_2, \ldots)$ where $x_k \varepsilon D_k$. For $x,y \varepsilon D^\infty$ define

$$d_\infty(x,y) = \sum_{k=1}^{\infty} 2^{-k} \left(\frac{d_k(x_k,y_k)}{1 + d_k(x_k,y_k)} \right)$$

Then (D^∞, d_∞) is a complete separable metric space. Now define the projection maps

$$r_k : D[0,\infty) \to D_k$$

by $r_k\big(x(t)\big) = x(t)$, $0 \le t \le k$, and let $\psi : D[0,\infty) \to D^\infty$ be defined by

$$\psi(x) = \{r_k(x),\ k \ge 1\}\ .$$

Then $\psi(D)$ is a closed subspace of D^∞ and $(\psi(D), d_\infty)$ is a complete separable metric space. To finish, define d on $D[0,\infty)$ by

$$d(x,y) = d_\infty\big(\psi(x),\psi(y)\big).$$

This is the metric that makes $D[0,\infty)$ a complete separable metric space and under which convergence is equivalent to convergence on $D[0,k]$ for all k. For a detailed treatment of convergence in this space, the reader is referred to Lindvall [29].

We now define determining and convergence-determining classes:

Def. 2.14: $\mathcal{U}\subset\mathcal{S}$ is a determining class if two probability measures P and Q on $\mathcal{S}$ such that $P = Q$ on $\mathcal{U}$ implies that $P \equiv Q$. $\mathcal{U}$ is a convergence determining class if for any sequence of probability measures on $\mathcal{S}, \{P_n\}_{n=0}^{\infty}$,

$$P_n[A] \to P_o[A] \quad \text{for all} \quad A \in \mathcal{U} \ni P_o[\partial A] = 0$$

implies that $P_n \Rightarrow P_o$.

A convergence determining class is always a determining class. In many cases the reverse is also true. For instance, in the cases of R^1 and R^k discussed above, and even in sequence space R^∞, the finite dimensional rectangles are both determining and convergence determining. Therefore to establish weak convergence it is sufficient to establish convergence of the finite dimensional distributions which can be done, for example, using characteristic functions. The essential difficulty in $D[0,\infty)$ is that the finite dimensional rectangles are not convergence determining. Therefore, convergence of the finite dimensional distributions is necessary but not sufficient for weak convergence.

Convergence of the finite dimensional distributions along with the notion of tightness is necessary and sufficient for weak convergence in $D[0,\infty)$ and most of the classical proofs use this argument. However, tightness is usually quite difficult to establish in specific cases. In our proof of weak convergence for the C_t process in Chapter VI we are able to avoid tightness arguments by considering an embedded renewal process and an appeal to a technique similar to that used by Durrett and Resnick [20] in discussing weak convergence with

random indices. The major usefulness of weak convergence lies in the continuous mapping theorem, which we now describe.

Theorem 2.10: Suppose that X_n, $n \geq 0$ are random elements of $(S,\mathcal{S})$ defined on $(\Omega,\mathcal{B},P)$. Suppose that $h:(S,\mathcal{S},\rho) \to (S',\mathcal{S}',\rho')$, i.e., h is a measurable map from S into another metric space S'. Let $\text{Disc } h = \{x \varepsilon S | h$ is discontinuous at $x\}$. If $P[X_0 \varepsilon \text{Disc } h] = 0$ and $X_n \Rightarrow X_0$ than $h(X_n) \Rightarrow h(X_0)$.

By considering useful maps onto other metric spaces, then, weak convergence of derived processes can be easily obtained from the basic weak convergence. For example, consider the mapping from $D[0,\infty)$ into $D^2[0,\infty)$ defined by

$$h_t(x) = \left(\bigvee_{0\leq u\leq t} x(u),\ \bigwedge_{0\leq u\leq t} x(u) \right) .$$

This is a continuous mapping from $D[0,\infty)$ into $D^2[0,\infty)$, so that if we can establish that $x_n(\cdot) \Rightarrow x_0(\cdot)$ in $D[0,\infty)$, then it follows that

$$\left(\bigvee_{0\leq u\leq t} x_n(u),\ \bigwedge_{0\leq u\leq t} x_n(u) \right) \Rightarrow \left(\bigvee_{0\leq u\leq t} x_0(u),\ \bigwedge_{0\leq u\leq t} x_0(u) \right) .$$

Now consider the continuous mapping from $D^2[0,\infty)$ into $D[0,\infty)$ defined by $h(x,y) = x-y$, and we can establish that

$$\bigvee_{0\leq u\leq t} x_n(u) - \bigwedge_{0\leq u\leq t} x_n(u) \Rightarrow \bigvee_{0\leq u\leq t} x_0(u) - \bigwedge_{0\leq u\leq t} x_0(u)$$

so that the range function of the random elements $x_n(\cdot)$ will converge weakly to the range function of the limiting random element $x_0(\cdot)$.

CHAPTER III

ANALYSES OF MOMENTS AND AUTOCORRELATION FUNCTION FOR THE DOUBLY INFINITE DAM

The doubly infinite dam is of interest for preliminary sizing studies, as has been discussed earlier. Before any useful information can be obtained from an analysis of, say, the range, a particular model must be entertained. Not only must one model be selected out of various competing models, but also once a model is selected there may be several parameters which must be estimated in some fashion. For example, the Markovian models considered in this investigation contain, in the general formulation, the following parameters:

m states $\mu_1,\ldots,\mu_m$

(m-1)m jump probabilities π_{ij}

m mean holding times $\lambda_1^{-1},\ldots,\lambda_m^{-1}$.

This is a total of $m(m+1)$ parameters which, if m is only slightly large makes the model difficult to work with. The two-state case contains two rates and two mean holding times for a total of four parameters, and the symmetric case reduces to two parameters. Although these special cases undoubtedly oversimplify the situation, nevertheless they contain a manageable number of parameters from which exact expressions of quantities of interest may be explicitly obtained.

In order to both check the fit of the model and perform an initial estimate of the parameters it is desirable to know the theoretical moments, or in this case, moment functions for the model. In particular the autocorrelation function is of benefit in the fitting of stochastic

models, since it can be matched with the sample autocorrelation function of the data, which can be easily measured.

In this chapter we will investigate the following functions which we now define, for the general case (as far as possible), and the two-state case.

For a stochastic process Z_t, $t \geq 0$ with finite second moments, we define as usual the

<u>mean function</u> $m_t = \int uP[Z_t \varepsilon du]$

<u>second raw moment function</u> $m_t^{(2)} = \int u^2 P[Z_t \varepsilon du]$

<u>variance function</u> $\sigma_t^2 = m_t^{(2)} - m_t^2$

<u>cross-product function</u> $m_{st} = \int\int uvP[Z_t \varepsilon du, Z_s \varepsilon dv]$

<u>covariance function</u> $K_{t,s} = m_{st} - m_s m_t$

<u>autocorrelation function</u> $\rho_{t,s} = K_{t,s}/\sigma_t \sigma_s$

If the process Z_t is strictly stationary with finite variance, then clearly m_t and σ_t^2 will be constant and $\rho_{t,s}$ will be a function of $t-s$ only. If, on the other hand, these three conditions are met then Z_t is called <u>second-order stationary</u>.

We will now investigate as much as possible the above functions for the marginal process on the doubly infinite dam: X_t, the Markov chain, and $C_t = \int_0^t X_u du$, the unrestricted contents process.

3.1 <u>General Case</u>

3.1.1 <u>Markov Chain</u>

Suppose that the Markov chain X_t has, as generator,

$$Q_{ij} = \begin{cases} -\lambda_{ij}, i=j & i,j=1,\ldots,m \\ \lambda_i \pi_{ij}, \; i \neq j \end{cases}$$

and suppose that it is defined on states $\mu_1,\ldots,\mu_m$. Let

$\pi_i(t) = P[X_t=i]$ as before. In the subsequent formulae, we will drop the upper and lower indices on the summands, which will always be m and 1 respectively. We assume throughout that the eigenvalues of Q are distinct.

$$m_t = \sum_i \mu_i \pi_i\,(t) \tag{3.1}$$

$$\begin{aligned} m_{t,t+s} &= \sum_i \sum_j \mu_i \mu_j P[X_t = i,\ X_{t+s} = j] \\ &= \sum_i \sum_j \mu_i \mu_j P_{ij}(s) \pi_i(t) \quad \text{by the Markov property} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \sigma_t^2 &= \sum_i \mu_i^2 \pi_i(t) - \left[\sum_j \mu_i \pi_i(t)\right]^2 \\ &= \sum_{i \neq j}\sum \mu_i(\mu_i - \mu_j)\pi_i(t)\pi_j(t) \end{aligned} \tag{3.3}$$

For the covariance function we obtain, using (3.1) and (3.2)

$$K_{t,t+s} = \sum_i \sum_j \mu_i \mu_j P_{ij}(s)\pi_i(t) - \sum_i \mu_i \pi_i(t) \sum_j \mu_j \pi_j(t+s) \tag{3.4}$$

giving for the autocorrelation function

$$\rho_{t,t+s} = \frac{\sum_i \sum_j \mu_i \mu_j P_{ij}(s)\pi_i(t) - \sum_i \mu_i \pi_i(t) \sum_j \mu_j \pi_j(t+s)}{\left[\sum_{i \neq j}\sum \mu_i(\mu_i-\mu_j)\pi_i(t)\pi_j(t)\right]^{\frac{1}{2}}\left[\sum_{i \neq j}\sum \mu_i(\mu_i-\mu_j)\pi_i(t+s)\pi_j(t+s)\right]^{\frac{1}{2}}} \tag{3.5}$$

Now, when the Markov chain has its stationary distribution $\underset{\sim}{\pi}$, we see that $\rho_{t,t+s}$ is a function of s alone, given by

$$\rho_s = \frac{\sum_i \sum_j \mu_i \mu_j P_{ij}(s)\pi_i - \left(\sum_i \mu_i \pi_i\right)^2}{\sum_{i \neq j}\sum \mu_i(\mu_i-\mu_j)\pi_i\pi_j} \tag{3.6}$$

Using the expression for $P_{ij}(s)$ from section 2.3.2 this reduces to

$$\rho_s = \frac{\sum_i \sum_j \sum_{k=2}^{m} t_{ki} r_{kj} e^{\theta_k s} \pi_i \mu_i \mu_j}{\sum_i \sum_{\neq j} \mu_i (\mu_i - \mu_j) \pi_i \pi_j} \tag{3.7}$$

or

$$\rho_s = \sum_{k=2}^{m} C_k e^{\theta_k s} \tag{3.8}$$

where

$$C_k = \frac{\sum_i \sum_j t_{ki} r_{kj} \pi_i \mu_i \mu_j}{\sum\sum_{i \neq j} \mu_i (\mu_i - \mu_j) \pi_i \pi_j}$$

Since $Re(\theta_k) < 0$, this shows that $\rho_s \to 0$ as $s \to \infty$. This also shows that the rate of convergence is governed by the eigenvalue with largest real part.

3.1.2 Contents Process

We now consider the contents process $C_t = \int_0^t X_u du$. For this analysis we also assume that X_t is stationary. C_t is the net input (input-release) in $(0,t)$ for a doubly-infinite dam whose net rate of change of level at time u is X_u. This is the continuous time analog of the cumulative sums of a Markov chain $\{X_n, n=0,1,2,\ldots\}$ as considered by Odoom and Lloyd [36], Ali Khan and Gani [1] and others. Since the typical procedure is to observe the input at a discrete set of times $\{0,h,2h,\ldots\}$ it is useful to know, for modelling purposes, the mean and covariance function of the increment process

$$\Delta_h(t) = \int_{t-\frac{1}{2}h}^{t+\frac{1}{2}h} x(u)\, du = C\left(t + \frac{h}{2}\right) - C\left(t - \frac{h}{2}\right) \qquad (3.9)$$

These are the same increments treated by McNeil for a special two-state case where one of the states is zero. We will examine $\Delta_h(t)$ more generally. The relevance of $\Delta_h(t)$ is that by looking at $\rho_{t,t+h}$ for the $\Delta_h(t)$ variable, we will be examining the correlation between adjacent increments. This correlation will then provide an indication of the degree of approximation to our process by one with independent increments for which, of course, $\rho_{t,t+h} = 0$. For the rest of this section, the moment functions which we develop will be for the increments of the process, which we will denote by an argument of h in the function.

For the mean function we have

$$E\Delta_h(t) = E\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} x(u)\, du = \sum_i \mu_i \pi_i h$$

For the second raw moment function, we have

$$m_t^{(2)}(h) = E\,\Delta_h(t)^2 = E\left[\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} x(u)\, du\right]^2 =$$

$$\int_{t-\frac{h}{2}}^{t+\frac{h}{2}}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} E[x(u)x(\xi)]du\, d\xi$$

$$=2\int_{t-\frac{h}{2}}^{t+\frac{h}{2}}\int_{\xi}^{t+\frac{h}{2}} E[x(u)x(\xi)]\, du\, d\xi \qquad (3.10)$$

where we are now integrating over the half-rectangle in which $u > \xi$.

Using (3.2) and the results of section 2.3.2, we can write (3.10) as

$$m_t^{(2)}(h) = 2 \int_{t-\frac{h}{2}}^{t+\frac{h}{2}} \int_{\xi}^{t+\frac{h}{2}} \sum_i \sum_j \mu_i \mu_j \left[\pi_j + \sum_{k=2}^{m} t_{ki} r_{kj} e^{\theta_k (u-\xi)} \right] \pi_i \, du \, d\xi \tag{3.11}$$

After performing the indicated integrations, (3.11) becomes

$$m_t^{(2)}(k) = 2 \sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki} r_{kj}}{\theta_k^2} \left[e^{\theta_k h} \quad \theta_k h - 1 \right] \mu_i \mu_j \pi_i$$

$$+ \sum_i \sum_j \mu_i \mu_j \pi_i \pi_j \, h^2 \tag{3.12}$$

For the variance function we obtain

$$\sigma_t^2(h) = m_t^{(2)} - \left[\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} Ex(u) \, du \right]^2$$

$$= m_t^{(2)} - \left(\sum \mu_i \pi_i h \right)^2 \tag{3.13}$$

and, substituting from (3.12), we obtain

$$\sigma_t^2(h) = 2 \sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki} r_{kj}}{\theta_k^2} \left[e^{\theta_k h} - \theta_k h - 1 \right] \mu_i \mu_j \pi_i \tag{3.14}$$

For the covariance function we have

$$K_{t,t+h}(h) = \mathrm{Cov}\left[\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} x(u)\,du, \int_{t+\frac{h}{2}}^{t+\frac{3h}{2}} x(u)du\right]$$

$$= \int_{t+\frac{h}{2}}^{t+\frac{3h}{2}} \int_{t-\frac{h}{2}}^{t+\frac{h}{2}} \mathrm{Cov}[x(u)x(\xi)]\,d\xi\,du$$

and, substituting for $\mathrm{Cov}[\cdot,\cdot]$ from the numerator of (3.7), we have

$$K_{t,t+h}(h) = \int_{t-\frac{h}{2}}^{t+\frac{h}{2}}\left[\int_{t+\frac{h}{2}}^{t+\frac{3h}{2}} \sum_i \sum_j \sum_{k=2}^{m} t_{ki} r_{kj} e^{\theta_k(\xi-u)}\,d\xi\right]\mu_i\mu_j\pi_i du \tag{3.15}$$

The integration in (3.15) is straight-forward and we obtain finally,

$$K_{t,t+h}(h) = \sum_i \sum_j \sum_{k=2}^{m} t_{ki} r_{kj}\left(-\frac{1}{\theta_k^2}\right)\left(e^{\theta_k\left(t+\frac{3h}{2}\right)} - e^{\theta_k\left(t+\frac{h}{2}\right)}\right)$$

$$\cdot\left(e^{-\theta_k\left(t+\frac{h}{2}\right)} - e^{-\theta_k\left(t-\frac{h}{2}\right)}\right)\mu_i\mu_j\pi_i$$

Expanding the product makes the dependence on t disappear, and we have

$$K_{t,t+h}(h) = \sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki} r_{kj}}{\theta_k^2}\left[1 + e^{2\theta_k h} - 2e^{\theta_k h}\right]\mu_i\mu_j\pi_i \tag{3.16}$$

Since the variance of the increments is free of t, we can use equations (3.14) and (3.16) to write the correlation function as

$$\rho_{t,t+h}(h) = \frac{\sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki} r_{kj}}{\theta_k^2} \left[1+e^{2\theta_k h} -2 ^{\theta_k h} \right] \mu_i \mu_j \pi_i}{2\sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki} r_{kj}}{\theta_k^2} \left[e^{\theta_k h} - \theta_k h - 1 \right] \mu_i \mu_j \pi_i} \qquad (3.17)$$

McNeil notes that for his two-state model with $\mu_1 = a$ and $\mu_2 = 0$, $\rho_{t,t+h}$ is free of a. This becomes obvious from (3.17), since in this case there is only one summand and the $\mu_i \mu_j$ term becomes a^2, which cancels from the numerator and denominator.

We close this section by checking (3.17) for the McNeil model. For the McNeil model we have $m=2$, $\mu_1=a$ and $\mu_2=o$, whereas for a two-state model we have, as will be derived in the following section, $\pi_1 = \rho/\lambda+\rho$ and $\theta_2 = -(\lambda+\rho)$. Hence

$$\rho_{t,t+h}(h) = \frac{\left[1-e^{-(\lambda+\rho)} \right]^2}{2\left[(\lambda+\rho)h-1+e^{-(\lambda+\rho)h} \right]}$$

the same as found by McNeil.

3.2 Two-State Case

In the two-state case, we can obtain some simple expressions for the moment functions.

3.2.1 Markov Chain

In this case the generator matrix is

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \rho & -\rho \end{pmatrix}$$

which admits the spectral decomposition

$$Q = T\Lambda T^{-1}$$

with

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -(\lambda+\rho) \end{pmatrix}$$

and

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\lambda}{\sqrt{\lambda^2+\rho^2}} \\ \frac{1}{\sqrt{2}} & -\frac{\rho}{\sqrt{\lambda^2+\rho^2}} \end{pmatrix}$$

From which we get

$$T^{-1} = \frac{1}{\lambda+\rho} \begin{pmatrix} \sqrt{2}\,\rho & \sqrt{2}\,\lambda \\ \sqrt{\lambda^2+\rho^2} & -\sqrt{\lambda^2+\rho^2} \end{pmatrix}$$

The stationary distribution satisfies $\underset{\sim}{\pi}' Q = \underset{\sim}{0}'$ from which it follows that

$$\underset{\sim}{\pi}' = \left(\frac{\rho}{\rho+\lambda} , \frac{\lambda}{\rho+\lambda}\right)$$

From the spectral decomposition of Q, the transition matrix and the moment function are easily obtainable. Since this is the result of routine algebra, we merely state the results. The moment functions are calculated under the stationary distribution.

$$P(t) = (\lambda+\rho)^{-1}\left[\begin{pmatrix} \rho & \lambda \\ \rho & \lambda \end{pmatrix} + e^{-(\lambda+\rho)t} \begin{pmatrix} \lambda & -\lambda \\ -\rho & \rho \end{pmatrix}\right]$$

$$m_t = (\lambda+\rho)^{-1} (\rho\mu_1+\lambda\mu_2)$$

$$m_t^{(2)} = (\lambda+\rho)^{-1}(\rho\mu_1^2+\lambda\mu_2^2)$$

$$\sigma_t^2 = \rho\lambda(\lambda+\rho)^{-2}(\mu_1-\mu_2)^2$$

$$m_{t,t+s} = (\lambda+\rho)^{-2}\left[(\rho\mu_1+\lambda\mu_2)^2 + \lambda\rho(\mu_1-\mu_2)^2 e^{-(\lambda+\rho)s}\right]$$

$$K_s = \rho\lambda(\lambda+\rho)^{-2}(\mu_1-\mu_2)^2\ e^{-(\lambda+\rho)s}$$

$$\rho_s = e^{-(\lambda+\rho)s}$$

We note that all limiting values have an exponential rate of convergence with $(\lambda+\rho)$ as a multiplier of t. Thus λ and ρ both contribute equally to the order of convergence. In addition, note that ρ_s is free of μ_1 and μ_2.

3.2.2 Contents Process

For the contents process, when X_t is stationary, we obtain

$$m_t = \sum\mu_i\pi_i t = (\rho+\lambda)^{-1}\ (\rho\mu_1+\lambda\mu_2)t$$

$$m_t^{(2)} = \sum_i \sum_j \mu_i\mu_j\pi_i \int_0^t \left[\int_0^\xi P_{ij}(s)ds + \int_0^{t-\xi} P_{ij}(s)ds\right]d\xi$$

The integrations are straight-forward, and we obtain finally

$$m_t^{(2)} = m_t^2 - 2(\lambda+\rho)^{-4}\rho\lambda(\mu_1-\mu_2)^2\left[1-e^{-(\lambda+\rho)t}-(\lambda+\rho)t\right]$$

from which

$$\sigma_t^2 = 2(\lambda+\rho)^{-4}\rho\lambda(\mu_1-\mu_2)^2\left[e^{-(\lambda+\rho)t} + (\lambda+\rho)t - 1\right]$$

We notice from the above that the variance function is proportional to the square of the difference between the two states, and the large-t growth is approximately linear.

Rather than examining the correlation between arbitrary points of time, we again consider the correlation between increments, which is easily obtainable from eq. (3.17)

$$\rho_{t,t+h}(h) = \frac{\left[1 + e^{2\theta_2 h} - 2e^{\theta_2 h}\right] \sum_i \sum_j \frac{t_{2i} r_{2j}}{\theta_2^2} \mu_i \mu_j \pi_i}{2\left[-\theta_2 h + e^{\theta_2 h} - 1\right] \sum_i \sum_j \frac{t_{2i} r_{2j}}{\theta_2^2} \mu_i \mu_j \pi_i}$$

$$= \frac{\left[1 - e^{-(\lambda+\rho)h}\right]^2}{2\left[(\lambda+\rho)h - 1 + e^{-(\lambda+\rho)h}\right]} ,$$

the same as found by McNeil. Note that $\rho_{t,t+h}$ is free of the states. This is a slightly more general result than the one reported by McNeil where only one state was arbitrary.

CHAPTER IV

ANALYSIS OF THE RANGE IN THE DOUBLY INFINITE DAM

In this chapter we begin the range analysis for the unrestricted contents process $C_t = \int_0^t X_u du$ in the doubly infinite dam. Let T be an exponentially distributed random variable with mean s^{-1} and independent of the process $\{X_t\}$, and let

$$M_s = \sup\{C_u,\ 0 \le u \le T\}, \tag{4.1}$$

$$m_s = \inf\{C_u,\ 0 \le u \le T\} \tag{4.2}$$

where as before $\{X_t\}$ is a finite Markov chain with generator Q and

$$C_t = \int_0^t X_u du .$$

We shall determine the joint distribution of M_s and m_s which in principle determines the joint distribution of

$$M_t = \sup\{C_u,\ 0 \le u \le t\}, \tag{4.3}$$

$$m_t = \inf\{C_u,\ 0 \le u \le t\}, \tag{4.4}$$

and hence of the range

$$R_t = M_t - m_t .$$

A closed form expression for the distribution of R_t is difficult to obtain; however, we shall find an explicit expression for ER_t in the symmetric case and investigate its asymptotic behavior for large t.

4.1 Joint Distribution of M_s, m_s

Define

$$\theta(x,\mu_i) = P[x+M_s \le a,\ x+m_s \ge 0 \mid X_o = \mu_i]\ , \qquad 0 < x < a \tag{4.5}$$

Then, appealing to the strong Markov property of (X_t,C_t) as shown by Erickson [20-1] we can write the first jump equation for θ as

$$\theta(x,\mu_i) = \rho(x,\mu_i) + \sum_j \int_{(0\le x+\mu_i u\le a)} \theta(x+\mu_i u,\mu_j)\pi_{ij} e^{-\lambda_i u} \lambda_i du \tag{4.6}$$

where $\rho(x,\mu_i)$ represents the case in which there are no jumps. For $\rho(x,\mu_i)$, therefore, we can write

$$\rho(x,\mu_i) = \begin{cases} \displaystyle\int_0^{-x/\mu_i} e^{-\lambda_i u} se^{-su}\, du & (\mu_i < 0) \qquad (4.7) \\ \displaystyle\int_0^{\frac{a-x}{\mu_i}} e^{-\lambda_i u}\, se^{-su}\, du & (\mu_i > 0) \qquad (4.8) \end{cases}$$

Hence

$$\rho(x,\mu_i) = \begin{cases} \dfrac{s}{\lambda_i+s} \left[1 - e^{(\lambda_i+s)\ x/\mu_i}\right] & (\mu_i < 0) \qquad (4.9) \\ \dfrac{s}{\lambda_i+s} \left[1 - e^{(\lambda_i+s)\left(\frac{x-a}{\mu_i}\right)}\right] & (\mu_i > 0) \qquad (4.10) \end{cases}$$

θ is a bounded, measurable function on the state space of the bivariate process (X_t,C_t). Hence, as shown by Brockwell [12] the system of integral equations possess a unique bounded solution and $\theta(x,\mu_i)$ satisfies the Kolmogorov backward equations.

Writing,

$$\psi_i(x,s) = 1 - \theta(x,\mu_i) = 1 - P[M_s \leq a-x,\ m_s \geq -x \mid X_0 = \mu_i] \tag{4.11}$$

we find by differentiating (4.6) that

$$\frac{\partial}{\partial x}\,\underset{\sim}{\psi}(x,s) = D^{-1}(Q-sI)\,\underset{\sim}{\psi}(x,s) \qquad (0 < x < a) \tag{4.12}$$

where $\underset{\sim}{\psi}'(x,s) = \left(\psi_1(x,s),\ldots,\ \psi_m(x,s)\right)$

and $D = \mathrm{diag}(\mu_i)$

with boundary conditions

$$\begin{aligned} \psi_i(a,s) &= 1 \qquad (\mu_i > 0) \\ \psi_i(0,s) &= 1 \qquad (\mu_i < 0)\ . \end{aligned} \tag{4.13}$$

The system (4.12) has the solution

$$\underset{\sim}{\psi}(x,s) = \exp[-D^{-1}(Q-sI)x]\ \underset{\sim}{\psi}(0,s)$$

4.2 Explicit Solution for the Two-State Case

Suppose that X_t has generator

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \rho & -\rho \end{pmatrix},$$

and is defined on the states $\mu_1 > 0$ and $\mu_2 < 0$. In order to determine $\exp[-D^{-1}(Q-sI)x]$, we spectrally decompose $M = -D^{-1}(Q-sI)$. The eigenvalues are given by

$$\theta = \frac{1}{2}\left[\frac{\rho+s}{\mu_2} + \frac{\lambda+s}{\mu_1}\right] \pm \sqrt{\frac{1}{4}\left(\frac{\rho+s}{\mu_2} + \frac{\lambda+s}{\mu_1}\right)^2 - \frac{s(\lambda+\rho) + s^2}{\mu_1\mu_2}}$$

We note that, if we set

$$d = \frac{1}{2}\left(\frac{\rho+s}{\mu_2} + \frac{\lambda+s}{\mu_1}\right),$$

then

$$\theta = d \pm (|d| + \Delta), \quad \text{where} \quad \Delta > 0 .$$

Case 1: $d > 0$. Then $\theta_1 = d + d + \Delta = 2d + \Delta > 0$

$\theta_2 = d - d - \Delta = -\Delta < 0$

Case 2: $d < 0$. Then $\theta_1 = d - d + \Delta = \Delta > 0$

$\theta_2 = d - (-d) - \Delta = 2d - \Delta < 0$

So that in either case we have

$$\theta_1 > 0 \ , \ \theta_2 < 0.$$

The right and left eigenvectors can be taken to be

$$\underset{\sim}{t}_1' = (\lambda, \lambda+s-\theta_1\mu_1)$$

$$\underset{\sim}{t}_2' = (\lambda, \lambda+s-\theta_2\mu_1)$$

$$\underset{\sim}{r}_1 = \left(\mu_1\rho, \mu_2(\lambda+s-\theta_1\mu_1)\right)$$

$$\underset{\sim}{r}_2 = \left(\mu_1\rho, \mu_2(\lambda+s-\theta_2\mu_1)\right)$$

We now note that we can write $e^{Mx}\underset{\sim}{\nu}$, $\underset{\sim}{\nu}$ an arbitrary column vector, as:

$$e^{Mx}\underset{\sim}{\nu} = \underset{\sim}{t}_1\underset{\sim}{r}_1'\underset{\sim}{\nu}\, e^{\theta_1 x} + \underset{\sim}{t}_2\underset{\sim}{r}_2'\nu\, e^{\theta_2 x}$$

Hence

$$e^{Mx}\underset{\sim}{\nu} = \begin{pmatrix} A_1e^{\theta_1 x} + B_1e^{\theta_2 x} \\ A_2e^{\theta_1 x} + B_2e^{\theta_2 x} \end{pmatrix}$$

where

$$\frac{A_1}{A_2} = \frac{t_{11}}{t_{12}}, \text{ i.e., } \quad t_{12}A_1 = t_{11}A_2 \tag{4.14}$$

and

$$\frac{B_1}{B_2} = \frac{t_{21}}{t_{22}} \text{ , i.e., } t_{22}B_1 = t_{21}B_2 \text{ .}$$

With this in mind, we can write the system (4.12) as:

$$\begin{aligned} \psi_1(x,s) &= A_1 e^{\theta_1 x} + B_1 e^{\theta_2 x} \\ \psi_2(x,s) &= A_2 e^{\theta_1 x} + B_2 e^{\theta_2 x} \end{aligned} \qquad (4.15)$$

with auxiliary conditions for A_1, A_2, B_1, B_2 given by

$$\left.\begin{aligned} A_1 e^{\theta_1 a} + B_1 e^{\theta_2 a} &= 1 \\ A_2 + B_2 &= 1 \end{aligned}\right\} \text{ from the boundary conditions}$$

and

$$\left.\begin{aligned} (\lambda+s-\theta_1\mu_1)A_1 &= \lambda A_2 \\ (\lambda+s-\theta_2\mu_1)B_1 &= \lambda B_2 \end{aligned}\right\} \text{ from (4.14)}$$

Solving for A_i and B_i we obtain

$$A_1 = \frac{\lambda e^{\theta_2 a} - (\lambda+s-\theta_2\mu_1)}{e^{\theta_2 a}(\lambda+s-\theta_1\mu_1) - e^{\theta_1 a}(\lambda+s-\theta_2\mu_1)} \qquad (4.16)$$

$$A_2 = \left[\frac{\lambda+s-\theta_1\mu_1}{\lambda}\right] A_1$$

$$B_1 = \frac{\lambda e^{\theta_1 a} - (\lambda+s-\theta_1\mu_1)}{e^{\theta_1 a}(\lambda+s-\theta_2\mu_1) - e^{\theta_2 a}(\lambda+s-\theta_1\mu_1)}$$

$$B_2 = \left[\frac{\lambda+s-\theta_2\mu_1}{\lambda}\right] B_1$$

The exact joint distribution of M_s and m_s are easily obtained from (4.15) and (4.16), from which the exact distribution of $R_s = M_s - m_s$,

the range to time T can be obtained in principle. An exact solution, however, is analytically intractable. Hence, rather than looking for the exact distribution of R_s, we will look instead for its expectation, since by linearity of the expectation operator, we can obtain this using knowledge only of the marginal distribution of M_s and m_s.

4.2.1 Marginal Distribution of M_s

In the definition of $\underset{\sim}{\psi}$, letting $y = a - x$, we have

$$\psi_i(y,s) = 1 - P[M_s \leq y,\ m_s \geq y - a | X_0 = \mu_i].$$

Therefore, the marginal distribution of M_s is given by

$$P[M_s > y] = \lim_{a\to\infty} \psi_i(y,s) \quad .$$

Since $\theta_1 > 0$ and $\theta_2 < 0$, and noting from 4.16 that $\lim_{a\to\infty} B_i$ exists and is finite, $i=1,2$, then from 4.15 it follows that

$$\lim_{a\to\infty} \psi_i(y,s) = \lim_{a\to\infty} A_i e^{\theta_1(a-y)} \quad , \ i = 1,2$$

and, after evaluating this limit we have

$$\psi_1(y,s) = e^{-\theta_1 y}$$

$$\psi_2(y,s) = \frac{\lambda+s-\theta_1\mu_1}{\lambda} e^{-\theta_1 y} \quad .$$

Therefore, if the initial rate is positive then M_s is exponentially distributed with parameter θ_1. If the initial rate is negative, then M_s has a truncated exponential distribution with parameter θ_1 and a mass at 0 of size

$$P_0 = \frac{\theta_1\mu_1-s}{\lambda}$$

Therefore it follows that

$$E[M_s|\mu_1] = \theta_1^{-1}$$

$$E[M_s|\mu_2] = (\theta_1\lambda)^{-1}(\lambda+s-\theta_1\mu_1)$$

4.2.2 Expected Value of R_s

By the law of total probability

$$ER_s = E(M_s-m_s) = \left[E(M_s|\mu_1) - E(m_s|\mu_1)\right]\left[P\ X(o) = \mu_1\right] + \left[E(M_s|\mu_2) - E(m_s|\mu_2)\right]\left[P\ X(o) = \mu_2\right]$$

However, from consideration of symmetry we have

$$E[m_s|\mu_1] = -\ E[M_s|-\mu_1]$$
$$E[m_s|\mu_2] = -\ E[M_s|-\mu_2]$$

Hence

$$E[M_s-m_s] = \left[E(M_s|\mu_1) + E(M_s|-\mu_1)\right] P[X(o) = \mu_1] + \left[E[M_s|\mu_2) + E(M_s|-\mu_2)\right] P[X(o) = \mu_2] \tag{4.17}$$

Stationary Case

We now evaluate ER_s when $X(t)$ has the stationary distribution, which in the two-state case is $\underset{\sim}{\pi}' = \left(\rho(\lambda+\rho)^{-1}\ \lambda(\lambda+\rho)^{-1}\right)$.

Substituting the appropriate values into (4.17) we obtain:

$$ER_s = (\lambda+\rho)^{-1}\left\{\theta_1^{-1}(\rho+\lambda+s-\theta_1\mu_1) + \tilde{\theta}_1^{-1}\left[\lambda+\rho\lambda^{-1}(\lambda+s+\tilde{\theta}_1\mu_2)\right]\right\} \tag{4.18}$$

where

$$\theta_1 = \frac{1}{2}\left[\frac{\rho+s}{\mu_2} + \frac{\lambda+s}{\mu_1}\right] + \sqrt{\frac{1}{4}\left(\frac{\rho+s}{\mu_2} + \frac{\lambda+s}{\mu_1}\right)^2 - \frac{s(\lambda+\rho)+s^2}{\mu_1\mu_2}}$$

$$\tilde{\theta}_1 = -\frac{1}{2}\left[\frac{\rho+s}{\mu_1} + \frac{\lambda+s}{\mu_2}\right] + \sqrt{\frac{1}{4}\left(\frac{\rho+s}{\mu_1} + \frac{\lambda+s}{\mu_2}\right)^2 - \frac{s(\lambda+\rho)+s^2}{\mu_1\mu_2}}$$

4.2.3 The Laplace Transform of ER_t

If $m_R(t) = ER_t$, then since

$$ER_s = \int_0^\infty m_R(t)se^{-st}dt ,$$

it follows that the Laplace transform with respect to time of ER_t, say $\hat{m}_R(s)$, is given by

$$\hat{m}_R(s) = s^{-1}ER_s .$$

Although the general two-state solution is easily obtained from (4.18) we will further restrict our attention to the symmetric case to perform an exact inversion:

Symmetric Case $\mu_1 = -\mu_2 = b$, $\lambda = p = a$

In this case we obtain after simple algebra that $\theta_1 = \tilde{\theta}_1 = \sqrt{\frac{2ab+s^2}{b^2}}$ and

$$\hat{m}_R(s) = ba^{-1}\left[\frac{2a+s}{\sqrt{2as+s^2}} - 1\right] \tag{4.19}$$

We can write $\hat{m}_R(s)$ as

$$\hat{m}_R(s) = ba^{-1}s^{-1}\left[(2as)^{-\frac{1}{2}}(2a+s)(1+s/2a)^{-\frac{1}{2}}-1\right].$$

Expanding $(1+s/2a)^{-\frac{1}{2}}$ in a Taylor series about o , we obtain

$$\left(1 + \frac{s}{2a}\right)^{-\frac{1}{2}} = 1 - \frac{1}{4a} s + \ldots + (-1)^n \frac{1.3\ldots(2n-1)}{a^n 4^n \cdot n!} s^n + \ldots$$

Hence, combining like powers, we have

$$(2a+s)\left(1 + \frac{s}{2a}\right)^{-\frac{1}{2}} = 2a + \left(1 - \frac{1}{2}\right) s + \left(-\frac{1}{4a} + \frac{1 \cdot 3}{4^2 a^2 2!} 2a\right) s^2$$

$$+ \ldots + \left[(-1)^{n-1} \frac{1 \cdot 3 \ldots (2n-3)}{4^{n-1} a^{n-1} (n-1)!}\right.$$

$$\left. + (-1)^n \frac{2 \cdot a \cdot 1 \cdot 3 \ldots (2n-1)}{4^n a^n n!}\right] s^n + \ldots$$

Noting that

$$\frac{1 \cdot 3 \ldots (2n-3)}{4^{n-1} a^{n-1} (n-1)!} - \frac{2 \cdot a \cdot 1 \cdot 3 \ \ldots \ (2n-1)}{4^n a^n n!} = \frac{2 \cdot 1 \cdot 3 \ \ldots \ (2n-3)}{4^n a^{n-1} n!}$$

we have

$$(2a+s)(1+s/2a)^{-\frac{1}{2}} = 2a + \frac{1}{2} s + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{2 \cdot 1 \cdot 3 \ldots (2n-3)}{4^n a^{n-1} \ n!} \ s^n$$

from which we get

$$\hat{m}_R(s) = \frac{1}{\sqrt{2}} b \ a^{-3/2} s^{-3/2} (2a+s)(1+s/2a)^{-\frac{1}{2}} - b a^{-1} s^{-1}$$

$$= \sqrt{2} \ b \ a^{-\frac{1}{2}} s^{-3/2} - b \ a^{-1} s^{-1} + \frac{\sqrt{2}}{4} b \ a^{-3/2} s^{-\frac{1}{2}}$$

$$+ \frac{\sqrt{2} \ ba^{-2}}{16} \sum_{j=0}^{\infty} (-1)^{j+1} \frac{1 \ 3 \ldots (2j+1)}{4^j (j+2)!} \left(\frac{s}{a}\right)^{\frac{2j+1}{2}} \tag{4.20}$$

valid for $0 < |s| < 2a$.

4.2.4 Explicit Inversion of Laplace Transform

Equation (4.20) gives an expansion for $\hat{m}_R(s)$ in increasing powers of s. Although such a form can not be used for term by term

inversion, it will be useful to derive an asymptotic expansion which will be discussed later. We now perform an expansion in terms of decreasing powers of s from which explicit inversion will be possible.

Returning to (4.19) we can write it as

$$\hat{m}_R(a,b,s) = ba^{-1}s^{-1}\left[\frac{2a+s}{\sqrt{2as+s^2}} - 1\right] = ba^{-1}s^{-1}\left[\left(1+\frac{2a}{s}\right)^{\frac{1}{2}} - 1\right]$$

Using the binomial expansion of $\left(1+\frac{2a}{s}\right)^{\frac{1}{2}}$, we obtain

$$\hat{m}_R(a,b,s) = ba^{-1}s^{-1}\left[\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\left(\frac{2a}{s}\right)^k - 1\right]$$

$$= b\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\frac{2^k a^{k-1}}{s^{k+1}} \tag{4.21}$$

valid for $\left|\frac{2a}{s}\right| < 1$, i.e., $|s| > 2a$.

Equation (4.21) provides a power series expansion for $\hat{m}_R(s)$ which is absolutely convergent in a neighborhood of infinity. In order to perform term-by-term inversion, we appeal to a theorem found in Doetsch [19] which we now state:

<u>Th</u>: When $\hat{f}(s)$ can be expanded in an absolutely convergent series for $|s| > R$ of the form

$$\hat{f}(s) = \sum_{n=0}^{\infty}\frac{a_n}{s^{\lambda_n}}$$

where the λ_n form an arbitrary increasing sequence of numbers $0 < \lambda_0 < \lambda_1 < \ldots \to \infty$, then the inversion can be executed term by term:

$$f(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\lambda_n - 1}}{\Gamma(\lambda_n)}$$

The series for $f(t)$ converges for all real and complex $t \neq 0$.

The conditions of the theorem are met by (4.21), with

$$a_n = \binom{\frac{1}{2}}{n} 2^n a^n \quad \text{and} \quad \lambda_n = n + 1 .$$

Therefore, we have for the inversion

$$m_R(a,b,t) = b \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \frac{2^k a^{k-1} t^k}{k!} \tag{4.22}$$

The series in (4.22) can be expressed in terms of a confluent hypergeometric series which we now define:

<u>Def</u>. The confluent hypergeometric series with parameters $\alpha \epsilon R$ and $\gamma \epsilon R$ is denoted by $F(\alpha,\gamma,x)$ and is given by

$$F(\alpha,\gamma,x) = \sum_{j=0}^{\infty} \frac{(\alpha)_j}{(\gamma)_j} \frac{x^j}{j!}$$

where $(\alpha)_j = \alpha(\alpha+1)\ldots(\alpha+j-1)$ for $j=1,2,\ldots$

$= 1$ for $j = 0$.

The confluent hypergeometric series converges for all x (Buchholz [15]), and has been tabulated in [27].

Returning to (4.22) we have

$$m_R(a,b,t) = \frac{b}{a} \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \frac{2^k a^k t^k}{k!} .$$

Hence,

$$m_R(a,b,t) = \frac{b}{a} \left[F(-\tfrac{1}{2},1,-2at)-1 \right] \tag{4.23}$$

The confluent hypergeometric function for a negative argument behaves like a very slowly converging alternating series. This makes it difficult to compute numerically. However, we can use Kummer's first

formula found in Buchholz to express it in terms of a hypergeometric function with positive argument. Kummer's first formula is

$$F(\alpha,\beta,x) = e^{x}F(\beta-\alpha,\beta,-x) \ .$$

Using this, we can re-write (4.23) as

$$m_R(a,b,t) = \frac{b}{a}\left[e^{-2at} F(3/2,1,2at)-1\right].$$

For numerical calculation on a computer this form is useful, for the confluent hypergeometric function is now a monotonic series which converges rather quickly, and the internal computer algorithms can be used to calculate the exponential term.

We present one last form for $m_R(a,b,t)$, using formula (14), page 7, of Buchholz:

$$m_R(a,b;t) = \frac{b2e^{-2at}}{a\sqrt{\pi}\Gamma(-\frac{1}{2})} \int_0^{2at} e^{\nu}\ \nu^{\frac{1}{2}}(2at-\nu)^{-3/2}\ d\nu \qquad (4.24)$$

4.2.5 Asymptotic Expansion of Laplace Transform

Equation (4.23) provides the exact mean of the range to time t in terms of a known function. However, the asymptotic behavior is not evident from this form. To study the asymptotic behavior, we make use of the technique of asymptotic expansion of the Laplace Transform, following Doetsch.

We first define an asymptotic expansion:

Def. A function $\phi(z)$ is said to have an asymptotic expansion $\sum_{\nu=0}^{\infty} \psi_\nu(z)$ as $z\to\infty$, written $\phi(z) \approx \sum_{\nu=0}^{\infty} \psi_\nu(z)$, iff

$$\phi(z) - \sum_{\nu=0}^{n} \psi_\nu(z) = o\left(\psi_n(z)\right) \quad \text{as} \quad z\to\infty \quad \text{for every} \quad n=1,2,\ldots \ .$$

Another way of stating this is to say that

$$\frac{\phi(z) - \sum_{\nu=0}^{n} \psi_\nu(z)}{\psi_n(z)} \to 0 \quad \text{as} \quad z\to\infty .$$

This says that the error $\phi(z) - \sum_{\nu=0}^{n} \psi_\nu(z)$ not only goes to zero, but it is of lower order of magnitude than the last term in the sum.

We now state a theorem found in Doetsch which can be used to obtain an asymptotic expansion for $m_R(t)$:

<u>Theorem</u>: If $\hat{f}(s)$ is the Laplace transform of $f(t)$, and $\hat{f}(s)$ can be expanded in a neighborhood of α_0 in an absolutely convergent power series with arbitrary (perhaps non-integral) exponents:

$$\hat{f}(s) = \sum_{\nu=0}^{\infty} C_\nu (s-\alpha_0)^{\lambda_\nu} , \; - N < \lambda_0 < \lambda_1 < \; \ldots \to\infty$$

(where N is a positive integer) then the following asymptotic expansion for $f(t)$ is valid for $t\to\infty$:

$$f(t) \simeq e^{\alpha_0 t} \sum_{\nu=0}^{\infty} \frac{C_\nu}{\Gamma(-\lambda_\nu)} t^{-\lambda_\nu - 1}$$

with $\frac{1}{\Gamma(-\lambda_\nu)} = 0$ if $\lambda_\nu = 0,1,2\ldots$.

We wish to apply this theorem to (4.20) with $\alpha_0 = 0$, but first we must remove the singularities at zero. To do this, define

$$\hat{f}(s) = \hat{m}_R(s) - \sqrt{2}\, b\, a^{-1/2} s^{-3/2} + b\, a^{-1} s^{-1} - \frac{\sqrt{2}}{4} b\, a^{-3/2} s^{-1/2}$$

Then

$$\hat{f}(s) = \frac{\sqrt{2} b a^{-2}}{16} \sum_{j=0}^{\infty} (-1)^{j+1} \frac{1\cdot3\ldots(2j+1)}{4^j (j+2)!} \left(\frac{s}{a}\right)^{\frac{2j+1}{2}} .$$

This is an absolutely convergent power series with

$$C_j = (-1)^{j+1} \frac{1\cdot 3\ldots(2j+1)a^{-\left(\frac{2j+1}{2}\right)}}{4j(j+2)!}$$

and $\lambda_j = \frac{2j+1}{2}$

Applying the theorem, then, we obtain the asymptotic expansion

$$f(t) = \frac{\sqrt{2}}{16}\left(\frac{b}{a}\right)\sum_{j=0}^{\infty} \frac{(-1)^{j+1} 1\cdot 3\ldots(2j+1)\,(at)^{-\frac{2j+3}{2}}}{4^j (j+2)!\,\Gamma\left(-\frac{2j+1}{2}\right)} \qquad (4.25)$$

We now invert $\hat{f}(s)$ term by term to obtain

$$f(t) = m_R(t) - b\sqrt{\frac{8}{a\pi}}\,t^{\frac{1}{2}} + \frac{b}{a} - \frac{3}{4}\sqrt{\frac{2}{\pi}}\,\frac{b}{a^{3/2}}\,t^{-\frac{1}{2}} \qquad (4.26)$$

Therefore, the asymptotic expansion of (4.26) is given by (4.25).

As an example of the application of this asymptotic expansion, we can take just the first term and obtain the limit

$$\lim_{t\to\infty} \frac{m_R(t) - b\sqrt{\frac{8}{a\pi}}\,t^{\frac{1}{2}} + \frac{b}{a} - \frac{3}{4}\sqrt{\frac{2}{\pi}}\,\frac{b}{a^{3/2}}\,t^{-\frac{1}{2}} - \frac{\sqrt{2}}{16}\left(\frac{b}{a}\right)\frac{(at)^{-3/2}}{2\Gamma(-1/2)}}{\frac{\sqrt{2}}{16}\left(\frac{b}{a}\right)\frac{(at)^{-3/2}}{2\Gamma(-1/2)}} = 0$$

or, equivalently,

$$\lim_{t\to\infty} \frac{m_R(t) - b\sqrt{\frac{8}{a\pi}}\,t^{\frac{1}{2}} + \frac{b}{a} - \frac{3}{4}\sqrt{\frac{2}{\pi}}\,\frac{b}{a^{3/2}}\,t^{-\frac{1}{2}}}{\frac{\sqrt{2}}{32}\,\frac{b}{a^{5/2}}\,\frac{t^{-3/2}}{\Gamma(-1/2)}} = 1$$

which says that

$$m_R(t) - \frac{b}{\sqrt{a}}\left[\sqrt{\frac{8}{\pi}}\,t^{\frac{1}{2}} - \frac{1}{\sqrt{a}}\right] - \frac{3}{4}\sqrt{\frac{2}{\pi}}\,\frac{b}{a^{3/2}}\,t^{-\frac{1}{2}} = 0\left(\frac{\sqrt{2}}{32}\,\frac{b}{a^{5/2}}\,\frac{t^{-3/2}}{\Gamma(-1/2)}\right).$$

In particular, since $t^{-\frac{1}{2}} \to 0$, then as $t \to \infty$

$$m_R(t) \sim \frac{b}{\sqrt{a}} \left[\sqrt{\frac{8}{\pi}}\, t^{\frac{1}{2}} \right] \quad . \tag{4.27}$$

Notice that if $b = \sqrt{a}$ then

$$m_R(t) \sim \sqrt{\frac{8}{\pi}}\, t^{\frac{1}{2}}$$

which is the value for the Wiener process. This is far from coincidental: In Chapter VI we will prove that under these conditions the process will in fact converge weakly to the Wiener process.

4.2.6 A Note on the Hurst Phenomenon

For the Wiener process it is well known that the maximum random variable has distribution

$$P[M_t \leq x] = 1 + 2\,\Phi(0,t;x),\ x \geq 0$$

where $\Phi(0,t;x)$ is the normal distribution function with mean 0 and variance t .

Hence we obtain for the mean, say EM_t

$$EM_t = \int_0^\infty \frac{2x}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}\, dx = \sqrt{\frac{2}{\pi}}\, t^{\frac{1}{2}}$$

Since the expected range is given by

$$ER_B(t) = EM_t - Em_t$$

and since for the Wiener process $Em_t = -EM_t$, it follows that

$$ER_B(t) = 2EM_t$$

or

$$ER_B(t) = \sqrt{\frac{8}{\pi}}\ t^{\frac{1}{2}}$$

By appealing to the continuous mapping theorem one can hope that this well-known result for the Wiener process will also hold asymptotically for any process which converges weakly to standard Brownian motion.

H. E. Hurst [26], in studying large amounts of data involving streamflows, rainfall, pressures, etc., found that ER_n appears to vary as n^{σ} with $.69 \leq \sigma \leq .80$. The average values of σ observed by Hurst was .72. Since his discrete time models for these processes were assumed to be made up of sums of independent radom variable, then the theoretical behavior of ER_n should have been as $n^{.5}$. This discrepancy became known as the Hurst phenomenon and was commented on in the introduction.

Although no completely satisfactory answer has been provided to the phenomenon, at least two theories have been suggested. The first is that the assumption of independence is not valid, and the observed growth of ER_n is the result of serial dependence in the series of summands. However, as noted by Moran [33], such a dependence would have to be of a very peculiar kind, since with all plausible models the large time behavior is approximated by an additive process, whose growth rate is $n^{0.5}$.

The second theory is that, due to dependence of the summands, the series studied by Hurst are not of sufficiently long duration to exhibit

the asymptotic behavior, and hence what he was seeing was the pre-asymptotic behavior. Several models have been suggested which exhibit this behavior in discrete time.

In our continuous time model, the analog of dependence of the summands is dependence of the increments of C_t. Asymptotically we have shown that ER_t behaves like $\sqrt{\frac{8}{\pi}}\ t^{\frac{1}{2}}$. However, it is of interest to investigate ER_t for finite t to see if the Hurst phenomenon appears pre-asymptotically.

In Figure 4.1 is a plot of $m_R(t)$ vs. t for both our Markov model with $a = b = 1$ and for Brownian motion. In Figure 4.2 is a plot on a logarithmic scale of $m_R(t)$ and $\sqrt{\frac{8}{\pi}}\ t^{.72}$ vs. t for $.1 \le t \le 2$. From this plot it appears that the rate of growth of $m_R(t)$ is closely approximated by a power of t, and within this range of time, the power of t is close to Hurst's average of 0.72.

The rapid convergence to the asymptotic value is apparently inconsistent with the length of the series studied by Hurst. However, by returning to equation 4.27, we note that taking a and b so that $b = a^{\frac{1}{2}}$ does not change the asymptotic growth rate. We now note from Eq. 4.24 that in this case,

$$m_R(a,b;t) = m_R(1,1;at) \quad .$$

Hence by choosing a sufficiently small, we equivalently perform a time expansion, so that the approach to the asymptotic value can be delayed as long as desired. Now, if our model is to display the Hurst phenomenon up to, say, $t = 2000$ (years), we would need to take $a = .001$. However, such a value of a implies that the correlation between successive yearly increments is, from Eq. (3.3),

$$\mathrm{Corr}[\Delta_1(t),\Delta_1(t+1)] = \frac{\left[1-e^{-.002}\right]^2}{2\left[.002-1+e^{-.002}\right]} = .99865$$

This value is much too high to be realistic. However, if we take $a = .1$, then we obtain

$$\mathrm{Corr}[\Delta_1(t),\Delta_1(t+1)] = .87720.$$

Thus a stream flow with a yearly correlation of this value and following this model would display Hurst behavior out to approximately 20 years.

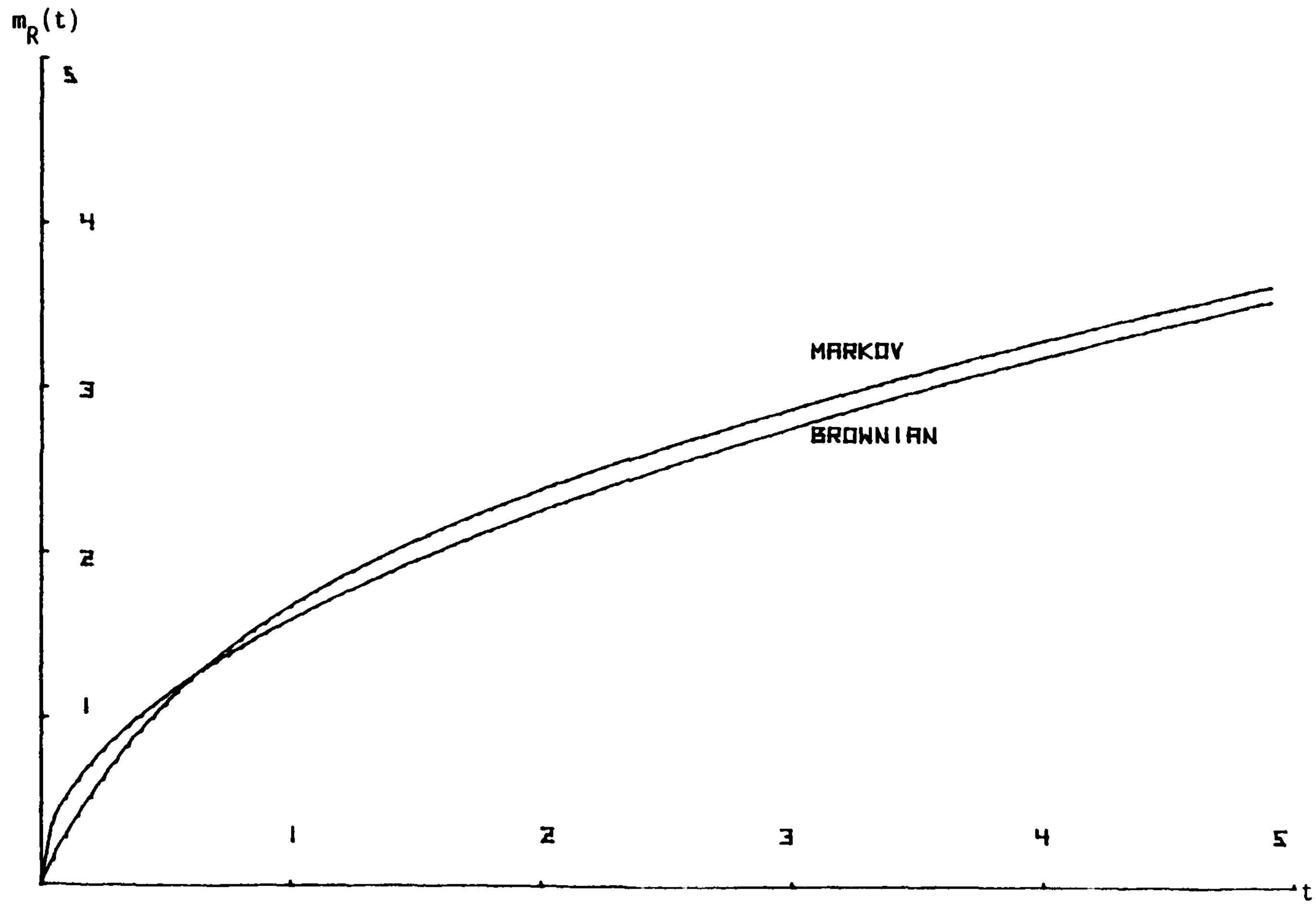


Figure 4.1 Expected range to time t versus t for Markov model and Brownian motion.

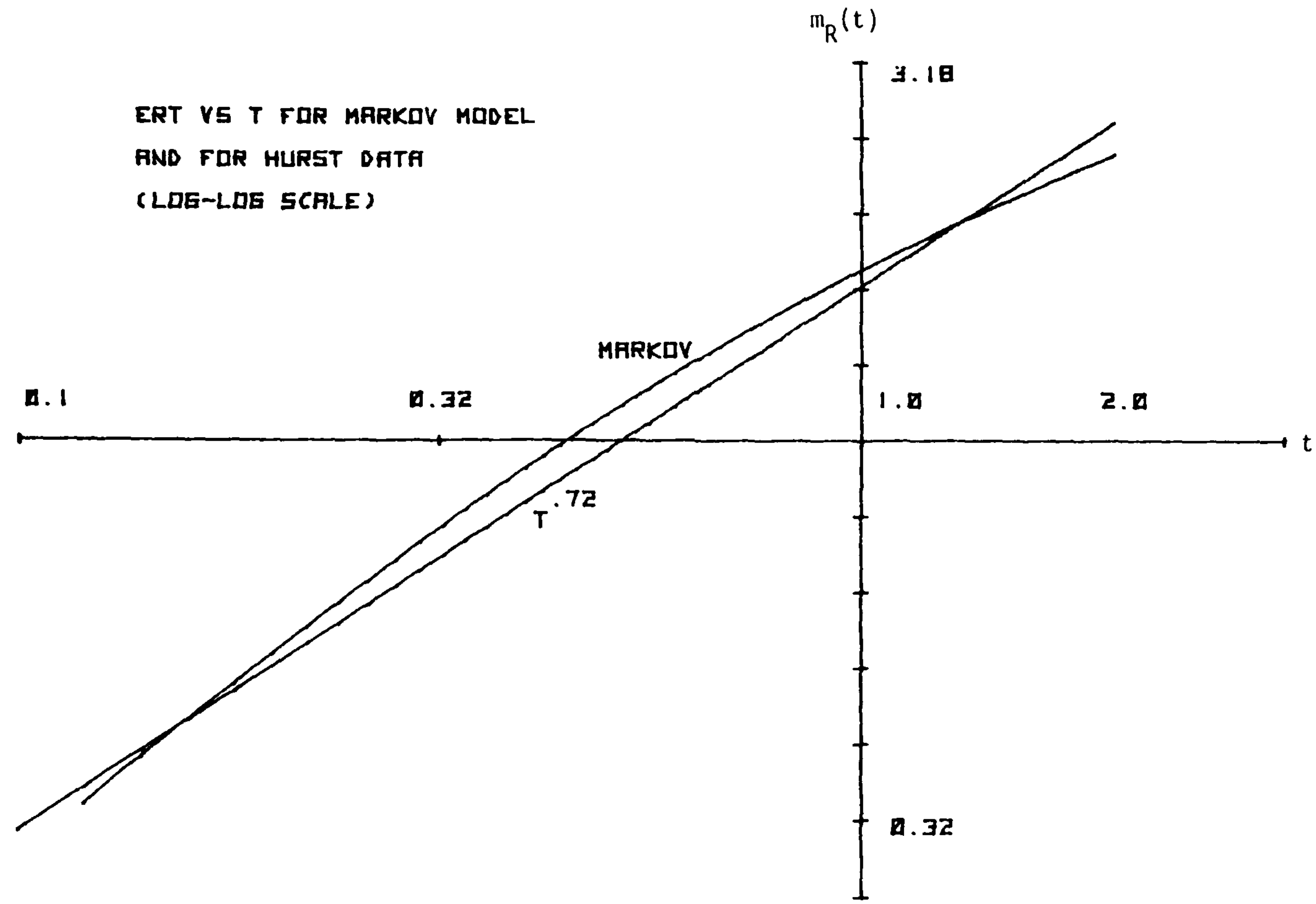


Figure 4.2 Comparison of growth rate of expected range for Markov model to average growth rate of Hurst data for $.1 \le t \le 2$.

CHAPTER V

INVARIANCE METHODS IN THE ANALYSIS OF SEMI-INFINITE AND FINITE DAMS

5.1 General Description

In this chapter we will develop the application of what we call in general "invariance methods" to the analysis of storage theory problems. These methods are well-known in the fields of astrophysics and particle physics, where the power of these techniques have been recognized and well utilized. Originally formulated by V. A. Ambarzumian [3] in 1944, the techniques were formally developed by S. Chandrasekhar [16] in a series of papers on radiative transfer through semi-infinite stellar atmospheres which appared in the Astrophysics Journal from 1944 to 1947. Subsequently Bellman and Kalaba in 1956 [5] refined the technique by applying it to finite regions in the field of particle transport, in which they called the technique 'invariant imbedding.' A good definition of invariance methods is given by M. Scott [40] who states that the method involves:

> generating a family of problems by means of a single parameter, where the basic properties of the system remain invariant under the generation of the family. The family then provides a means of advancing from one member, sometimes degenerate, to the solution of the original problem.

After providing a brief description of Chandrasekhar's and Bellman's application, we will establish what will be a natural application of these methods to storage theory. In particular we will take advantage of the power of the invariance methods to derive the Laplace transform of the 'wet period,' and from this obtain moments and the limiting probability of emptiness. This will lead us to a discussion of necessary and sufficient conditions for the recurrence of the contents

process. Although some of the results that we present in this section are not new, we will be able to derive them in a direct and simple fashion, avoiding the Kolmogorov differential equations which have previously been used to derive these results. In fact, in the case of the semi-infinite dams our equations are algebraic in nature.

5.1.1 Chandrasekhar's Principles of Invariance

In his application to astrophysics, Chandrasekhar was concerned with the transfer of radiation through a stellar atmosphere. In studying certain aspects of the radiation, such as the intensity, he considered the atmosphere to be stratified in parallel planes in which all of the pertinent physical properties are invariant over a plane. When considering a semi-infinite atmospheric region then, he utilized the fact that certain physical laws governing radiative transfer must be invariant to the addition or subtraction of layers of arbitrary optical thickness to or from the atmosphere. Through this 'principle of invariance' as he called it, he was able to increment the arguments in the equation governing the physical laws of radiative transfer while still maintaining the basic equations. These perturbations of the equations then led him to direct solutions for the physical quantities of interest.

In studying the time to emptiness in a semi-infinite dam, the essential similarities of the two problems become clear. In the case of the dam, the strong Markovian character of the bivariate process (C_t, X_t) makes it possible to consider the process as regenerative at the stopping times corresponding to first entrance into an upper semi-infinite region after traversing an incremental slab of arbitrary thickness. Thus random variables such as first passage times become

'invariant' to the addition or subtraction of these finite incremental slabs. Seen in this context, one notes the similarity between Chandrasekhar's Principles of Invariance in stellar atmospheres and the regenerative property of a Markov process. In fact, this entire discussion could be phrased in terms of regeneration. The essential difference, however, is that in the classical analysis of Markovian structures the principle of regeneration is used to establish Kolmogorov-type equations which are generally partial differential equations. These equations normally provide much more information about the process than is required for the study of particular aspects which become essentially boundary conditions for the Kolmogorov equations. For example, suppose that we are interested in the wet period for a semi-infinite dam: that is, the time elapsed from an initial exit from zero to a first return to zero. In the two-state case, if we let

$$T = \inf\{t > 0 : C_t = 0\}$$

then we may be interested in finding the Laplace transform of T,

$$\phi_s = E\left[e^{-sT} | X_0 = \mu_1 > 0\right]$$

Now, by setting up and solving the Kolmogorov backward equations, Brockwell was able to determine the Laplace transform of the time to first emptiness given an arbitrary initial level x. Thus if $\phi_{is}(x)$ represents this transform in the general case, then

$$\phi_{is}(x) = E\left[e^{-sT} | X_0 = \mu_i\right]$$

where $T = \inf\{t : t > 0 , C_t = -x\}$

However, the solution depends on $\phi_{is}(0)$, which in Brockwell's formulation was obtained from a rather complicated set of boundary conditions. Invariant imbedding leads, as we shall see, to a direct determination of ϕ_s as the solution of an initial value problem when the dam is finite and of an algebraic equation when the dam is infinite.

5.1.2 Bellman's Invariant Imbedding

As noted earlier, Bellman refined Chandrasekhar's Principles of Invariance and applied them to finite regions in particle transport theory. Two excellent comprehensive studies on these applications are those by Bellman, Kalaba, and Prestrud [6], in which they apply the technique to radiative transfer in slabs of finite thickness, and Bellman, Kalaba, Prestrud, and Kagiwada [7], in which they apply it to time dependent transport processes.

In these studies, Bellman called this technique "invariant imbedding," a name by which it has subsequently become known. The application to time dependent transport processes is particularly relevant to our study of the finite dam. Because the analogy to the finite dam problem is not obvious, we will discuss briefly the former process, and then draw the analogy.

The physical model in the transport process is of a one-dimensional rod of fixed length ℓ which is capable of transporting particles such as neutrons. The particles are allowed to travel to the right or to the left and can interact only with the fixed constituents of the rod. When a particle interacts with the rod, the old particle disappears and two new ones appear, one travelling to the right and one travelling to the left. All the particles travel with the same speed and their other physical properties are such that the particles are distinguishable

only by their direction. Suppose that we inject one particle at one end of the rod, and are interested in the total number of particles emanating from the other end. One approach to this problem would be to derive an equation for $u(x)$ = number of particles emanating from the end of the rod if we start with one particle at x , for $0 \leq x < \ell$. The desired quantity would then be given by $u(o)$. However, a more efficient approach would be to develop, if possible, an equation for $u(o)$ directly. This is precisely the approach used in invariant imbedding. The invariant imbedding technique is to embed the original rod of length ℓ into a rod of length $\ell + \delta\ell$, and develop a differential equation for $u(\ell)$. Whereas a differential equation for $u(x)$, $0 \leq x < \ell$ would lead to a two-point boundary value problem with boundaries at $x = 0$ and $x = \ell$, the differential equation for $u(o)$ is an initial value problem with initial value $\ell = 0$. This initial value is easily determined because of its degeneracy.

We now begin to see the analogy to the storage problem, which we will list in the form of a table.

Particle Transport	Storage Problem
1. Particles travel only to right or left with fixed velocity.	1. Dam contents only increase or decrease at fixed rates.
2. Do not need to know entire function $u(x)$, $0 \leq x < \ell$, but only $u(o)$.	2. Do not need to know $\phi_{is}(x)$ for all $0 \leq x \leq a$, but only $\phi_s(o)$
3. Differential equations for $u(x)$ lead to a two-point boundary value problem, whereas differential equations for $u(o)$ give an initial value problem.	3. Differential equations for $\phi_{is}(x)$ lead to a two-point boundary value problem, whereas differential equations for $\phi_s(o)$ gives an initial value problem.

Table 5.1 Analog between Particle Transport and Storage Problem.

5.2 Applications of Invariance to First Passage Times

We will now use invariance techniques to study the distribution of first passage times. In a later section we will discuss necessary and sufficient conditions for these times to be finite.

5.2.1 First Passage Times for the Semi-Infinite Dam

5.2.1.1 General Case

We define the following functions related to first passage times:

(a) The Laplace transform of the first passage time to level y :

Let $\tau = \inf\{u : C_u = y\}$, (5.1)

and define

$$T^{+}_{ij}(y) = E\left[e^{-s\tau} I_{\{X_\tau = \mu_j\}} | X_0 = \mu_i\right] \qquad (5.2)$$

for $\mu_i > 0$, $\mu_j > 0$, $y > 0$ and

$$T^{-}_{ij}(y) = E\left[e^{-s\tau} I_{\{X_\tau = \mu_j\}} | X_0 = \mu_i\right] \qquad (5.3)$$

for $\mu_i < 0$, $\mu_j < 0$, $y < 0$.

If there are p positive states and $m-p$ negative states, let T^{+} and T^{-} represent the corresponding matrices with entries T^{+}_{ij} and T^{-}_{ij} , of dimension $p \times p$ and $(m-p) \times (m-p)$.

(b) The Laplace transform of the first return time to level zero:

For $\mu_i > 0$, $\mu_j < 0$, let

$$R^{+-}_{ij} = E\left[e^{-sT} I_{\{X_T = \mu_j\}} | X_0 = \mu_i\right] \qquad (5.4)$$

where

$$T = \inf\{t > 0 : C_t = 0\} \tag{5.5}$$

Let R^{+-} be the $px(m-p)$ matrix $[R_{ij}^{+-}]$.

For $\mu_i < 0$, $\mu_j > 0$, the right side of (5.4) will be denoted by R_{ij}^{-+} and the $(m-p)xp$ matrix $[R_{ij}^{-+}]$ by R^{-+} .

(c) The Laplace transform of the first return time to level zero with *no* prior passage through a given level.

For $\mu_i > 0$, $\mu_j < 0$, and $y \geq 0$, let

$$R_{ij}^{+}(y) = E\left[e^{-sT} I_{\{X_T = \mu_j\}} I_{\{C(t)\neq y,\ 0\leq t\leq T\}} \mid X_0 = \mu_i\right] \tag{5.6}$$

For $\mu_i < 0$, $\mu_j > 0$, let

$$R_{ij}^{-}(y) = E\left[e^{-sT} I_{\{X_T = \mu_j\}} I_{\{C(t)\neq -y,\ 0\leq t\leq T\}} \mid X_0 = \mu_i\right] \tag{5.7}$$

Let $R^{+}(y)$ be the $px(m-p)$ matrix $[R_{ij}^{+}(y)]$ and let $R^{-}(y)$ be the $(m-p)xp$ matrix $[R_{ij}^{-}(y)]$.

The Laplace transform of the first return time of the content of a topless dam to zero given that $X_0 = \mu_i$ is

$$\sum_{j:\mu_j<0} R_{ij}^{+-} ,$$

where R_{ij}^{+-} is defined by (5.4). We now use an imbedding argument to determine R^{+-} . The time at which a sample path, starting at the zero level, crosses into the upper region (y,∞) is a stopping time, and the strong Markov property of (X_t, C_t) may be applied at these times. Using this idea to obtain the desired first return time to zero, we decompose R according to the number of reflections of the sample path in the slab $(0,y)$ to obtain the relation

$$
\begin{aligned}
R^{+-} = R^{+}(y) &+ T^{+}(y)R^{+-}T^{-}(y) \\
&+ T^{+}(y)R^{+-}R^{-}(y)R^{+-}T^{-}(y) \\
&+ T^{+}(y)R^{+-}R^{-}(y)R^{+-}R^{-}(y)R^{+-}T^{-}(y) \\
&+ \ldots
\end{aligned} \tag{5.8}
$$

$$
= R^{+-}(y) + T^{+}(y)\, R^{+-}[I - R^{-}(y)R^{+-}]^{-1}T^{-}(y) \tag{5.9}
$$

We now consider the limiting behavior of the matrices in (5.9) for small y . First, note that

$$
T^{+}_{ii}(y) = \left(1-\lambda_i \frac{y}{\mu_i}\right) e^{-s\frac{y}{\mu_i}} + o(y) \quad \text{as} \quad y \to 0 \tag{5.10}
$$

and, expanding the exponential,

$$
T^{+}_{ii}(y) = 1 - \frac{y}{\mu_i}(\lambda_i + s) + o(y) \tag{5.11}
$$

Similarly,

$$
T^{+}_{ij}(y) = \lambda_i \pi_{ij} \frac{y}{\mu_i}\; e^{-s\frac{y}{\mu_i}} + o(y)\ , \quad i \neq j \tag{5.12}
$$

$$
= \lambda_i \pi_{ij} \frac{y}{\mu_i} + o(y) \tag{5.13}
$$

Therefore,

$$
T^{+}(y) = I - y\left[D^{-1}(sI-Q)\right]_{++} + o(y) \tag{5.14}
$$

where, for an $m \times m$ matrix M we define $[M]_{++}$, $[M]_{+-}$, $[M]_{-+}$, and $[M]_{--}$ to be the partitions defined by

$$
M = \begin{array}{cc} \left[\begin{array}{cc} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{array}\right] & \begin{array}{c} p \\ (m-p) \end{array} \\ \begin{array}{cc} p & (m-p) \end{array} & \end{array} \tag{5.15}
$$

Through parallel reasoning,

$$T^{-}(y) = I + y\left[D^{-1}(sI-Q)\right]_{--} + o(y) \tag{5.16}$$

For the R functions, it can be shown that

$$R^{+}(y) = y[D^{-1}Q]_{+-} + o(y) \tag{5.17}$$

and

$$R^{-}(y) = -y[D^{-1}Q]_{-+} + o(y) \tag{5.18}$$

Substituting (5.14), (5.16), (5.17), and (5.18) into (5.9) we obtain

$$R^{+-} = y(D^{-1}Q)_{+-} + \left(I - y\left[D^{-1}(sI-Q)\right]_{++}\right)$$

$$+ R^{+-}[I+y(D^{-}Q)_{-+}R^{+-}]^{-1}\left(I+y\left[D^{-1}(sI-Q)\right]_{--}\right) + o(y) \tag{5.19}$$

Since

$$\left[I+y(D^{-1}Q)_{-+}R^{+-}\right]^{-1} = I-y(D^{-1}Q)_{-+}R^{+-} + o(y), \tag{5.20}$$

then

$$R^{+-} = y(D^{-1}Q)_{+-} + R^{+-} - y\left[D^{-1}(sI-Q)\right]_{++}R^{+-}$$

$$- R^{+-}y(D^{-1}Q)_{-+}R^{+-} + R^{+-}y\left[D^{-1}(sI-Q)\right]_{--} + o(y) \tag{5.21}$$

from which we obtain, by taking the limit as $y \to 0$,

$$R^{+-}(D^{-1}Q)_{-+}R^{+-} + \left[D^{-1}(sI-Q)\right]_{++}R^{+-} - R^{+-}\left[D^{-1}(sI-Q)\right]_{--}$$

$$- (D^{-1}Q)_{+-} = 0 \tag{5.22}$$

5.2.1.2 Two-state Case - Laplace Transform of Wet Period

For the two-state case we remarked earlier that the first passage time corresponds to the wet period in the dam. In this

case we set $\phi_s = R_{12}^{+-}$ in (5.4) and, making the appropriate substitutions, equation (5.22) reduces to

$$\rho\mu_2^{-1}\ \phi_s^2 - [\rho\mu_2^{-1} - \lambda\mu_1^{-1} - s(\mu_1^{-1}-\mu_2^{-1})]\ \phi_s - \lambda\ \mu_1^{-1} = 0 \tag{5.23}$$

from which we obtain the two solutions

$$\phi_s = \mu_2(2\rho)^{-1}\left[A \pm (A^2+4\lambda\rho\mu_1^{-1}\mu_2^{-1})^{\frac{1}{2}}\right] \tag{5.24}$$

where

$$A = s(\mu_2^{-1}-\mu_1^{-1}) + \rho\mu_2^{-1}-\lambda\mu_1^{-1}$$

We must now select the correct solution. Since

$$A^2 \to \infty \quad \text{as} \quad s \to \infty\ , \quad \text{then} \quad \phi_s$$

is unbounded for the positive solution, and so we conclude that

$$\phi_s = \mu_2(2\rho)^{-1}\left[A - (A^2+4\lambda\rho\mu_1^{-1}\ \mu_2^{-1})^{\frac{1}{2}}\right]. \tag{5.25}$$

We now investigate conditions under which the first return to zero, T , is finite. First note that

$$\phi_0 = \mu_2(2\rho)^{-1}\left[\rho\mu_2^{-1} - \lambda\mu_1^{-1} - |\rho\mu_2^{-1} + \lambda\mu_1^{-1}|\right] \tag{5.26}$$

and

$$\rho\mu_2^{-1} + \lambda\mu_1^{-1} = m\left(\frac{\rho+\lambda}{\mu_1\mu_2}\right),$$

where $m = EX_t$ when X_t has its stationary distribution. We call m the <u>drift</u> of the process. Therefore,

$$\phi_0 = \begin{cases} 1 & \text{if} \quad m \leq 0 \\ -\dfrac{\lambda}{\rho}\ \dfrac{\mu_2}{\mu_1} & \text{if} \quad m > 0 \end{cases} \tag{5.27}$$

Since $\phi_0 = P[T < \infty]$, then from (5.27) it follows that for the topless dam, a non-positive drift implies that $P[T < \infty] = 1$, whereas a positive drift implies that $P[T < \infty] = -\frac{\lambda\mu_2}{\rho\mu_1}$.

5.2.1.3 Distribution of Wet Period for the Symmetric Case

If we set $\mu_1 = b, \mu_2 = -b, \lambda = \rho = a$, then the Laplace transform equation becomes

$$\frac{a}{b}\phi_s^2 - \left[\frac{2a}{b} + \frac{2s}{b}\right]\phi_s + \frac{a}{b} = 0$$

or

$$\phi_s^2 - 2\left(1 + \frac{s}{a}\right)\phi_s + 1 = 0 \tag{5.28}$$

From this equation we see the rather surprising result that the time to first emptiness does not depend on the states on which the Markov chain is defined.

Solving the quadratic equation, we obtain

$$\phi_s = 1 + \frac{s}{a} - \left(\frac{s^2}{a^2} + \frac{2s}{a}\right)^{\frac{1}{2}} \tag{5.29}$$

which can be written in terms of $1/s$ as

$$\phi_s = \frac{s}{a}\left[1 - \left(1 + \frac{2a}{s}\right)^{\frac{1}{2}}\right] + 1 \tag{5.30}$$

Expanding $\left(1 + \frac{2a}{s}\right)$ in its binomial expansion, we obtain

$$\phi_s = \frac{s}{a}\left[1 - \sum_0^\infty \binom{\frac{1}{2}}{k}\left(\frac{2a}{s}\right)^k\right] + 1 \tag{5.31}$$

Performing a term-by-term inversion of (5.31), we obtain the density of T , say $f_T(x)$, $x \geq 0$ as

$$f_T(x) = 2\left[\frac{(-\frac{1}{2})(\frac{1}{2})}{1\cdot 2} (-2a) + \frac{(-\frac{1}{2})(\frac{1}{2})(3/2)}{1\cdot 2\cdot 3} \frac{(-2a)^2 x}{1!} + \frac{(-\frac{1}{2})(\frac{1}{2})(3/2)(5/2)}{1\cdot 2\cdot 3\cdot 4} \frac{(-2a)^3}{2!} x^2 + \ldots \right] \tag{5.32}$$

We can write this in terms of a confluent hypergeometric series as

$$f_T(x) = \frac{a}{2} F(3/2,\ 3,\ -2ax) \tag{5.33}$$

For the distribution function, say

$$G_T(x) = P[T \leq x] = \int_0^x f_T(u)du$$

we can integrate the power series term by term to obtain the series

$$G_T(x) = \frac{a}{2}\left[x + \frac{3/2}{3} \frac{(-2a)}{1!\cdot 2} x^2 + \frac{(3/2)(5/2)}{3\cdot 4} \frac{(-2a)^2}{2!\cdot 3} x^3 + \ldots \right] \tag{5.34}$$

5.2.1.4 Time Between Overflows for the Semi-Infinite Bottomless Dam

We now consider a semi-infinite dam with a top but no bottom. This may in fact be a more realistic model than the topless dam, since actual dams are constructed and operated in such a way that the probability of overflow is higher than the probability of emptiness.

The analysis of the bottomless dam is directly analogous to the analysis of the topless dam. Therefore, we will be brief in the following derivations. In the bottomless dam, if we measure the contents relative to the top of the dam, which we can set equal to zero without loss of generality, then for the contents process we have

$$C_t^B = \int_0^t X^*(u)du \tag{5.35}$$

where

$$X^*(u) = \begin{cases} 0 & \text{if } C_u^B = 0 \text{ and } X_u > 0 \\ X_u & \text{otherwise.} \end{cases}$$

For the bottomless dam, $C_t^B = 0$ represents an overflow condition. Let ϕ_s be the Laplace transform of the time to first overflow, given a starting condition, $X_0 = \mu_2$. Then by considerations of symmetry we can conclude that ϕ_s satisfies equation (5.25) with the parameters λ, ρ, and μ_1, μ_2 interchanged. Therefore ϕ_s satisfies

$$\lambda\mu_1^{-1}\phi_s^2 - \left[\lambda\mu_1^{-1} - \rho\mu_2^{-1} - s(\mu_2^{-1}-\mu_1^{-1})\right]\phi_s - \rho\mu_2^{-1} = 0 \tag{5.36}$$

from which it follows that

$$\phi_s = \mu_1(2\lambda)^{-1}\left[A-(A^2+4\lambda\rho\mu_1^{-1}\ \mu_2^{-1})^{\frac{1}{2}}\right] \tag{5.37}$$

where

$$A = s(\mu_1^{-1}-\mu_2^{-1}) + \lambda\mu_1^{-1} - \rho\mu_2^{-1}$$

The explicit inversion for the symmetric case is identical, since in that case the two exchanged parameters are equal. Therefore, the distribution of the time between overflows in the bottomless dam is equal to the distribution of the wet period in the topless dam, as expected.

By similar reasoning to that used to establish (5.27), we conclude that for the bottomless dam, a non-negative drift implies that $P[T < \infty] = 1$, whereas a negative drift implies that

$$P[T < \infty] = -\frac{\rho\mu_1}{\lambda\mu_2}$$

5.2.2 First Passage Times for the Finite Dam

5.2.2.1 General Case

We now turn our attention to the finite dam with an upper boundary at a. In the same vein as (5.4), we define

$$R_{ij}^{+-}(a) = E\left[e^{-sT} I_{\{X_T = \mu_j\}} \mid X_0 = \mu_i\right] \tag{5.38}$$

with $\{C_t\}$ replaced by $\{C_t^F\}$, the content process for the finite dam where T is given as in (5.5). Let $R(a)$ be the $p\mathrm{x}(m-p)$ matrix $\left[R_{ij}^{+-}(a)\right]$.

Using the same matrices defined for the semi-infinite dam, and using the same argument of decomposing $R^{+-}(a)$ according to the number of reflections of the sample path in the slab $(0,y)$, $0 < y < a$, we can write the relations

$$\begin{aligned} R^{+-}(a) &= R^{+}(y) + \sum_{i=0}^{\infty} T^{+}(y)R^{+-}(a-y)\left[R^{-}(y)R^{+-}(a-y)\right]^{i} T^{-}(y) \\ &= R^{+}(y) + T^{+}(y)R^{+-}(a-y)\left[I-R^{-}(y)R^{+-}(a-y)\right]^{-1}T^{-}(y) \end{aligned} \tag{5.39}$$

Using the relations (5.17) - (5.20), we can write

$$\begin{aligned} R^{+-}(a) &= y(D^{-1}Q)_{+-} + \left(I-y\left[D^{-1}(sI-Q)\right]_{++}\right) \\ &\cdot R^{+-}(a-y)\left[I-y(D^{-1}Q)_{-+}R^{+-}(a-y)\right]\left(I+y\left[D^{-1}(sI-Q)\right]_{--}\right) + o(y) \end{aligned} \tag{5.40}$$

so that

$$\begin{aligned} \frac{R^{+-}(a) - R^{+-}(a-y)}{y} &= - R^{+-}(a-y)(D^{-1}Q)_{-+}R^{+-}(a-y) \\ &\quad -\left[D^{-1}(sI-Q)\right]_{++}R^{+-}(a-y) + R^{+-}(a-y)\left[D^{-1}(sI-Q)\right]_{--} \\ &\quad + (D^{-1}Q)_{+-} + \frac{o(y)}{y} \end{aligned} \tag{5.41}$$

Taking the limit as $y \to 0$ we obtain

$$\frac{dR^{+-}(a)}{da} + R^{+-}(a)(D^{-1}Q)_{-+}R^{+-}(a) + [D^{-1}(sI-Q)]_{++}R^{+-}(a)$$
$$- R^{+-}(a)[D^{-1}(sI-Q)]_{--} - (D^{-1}Q)_{+-} = 0 \qquad (5.4.)$$

The initial condition is

$$R^{+-}(0) = Ee^{-s\tau^*_{ij}}$$

where τ^*_{ij} is the first passage time from state i to state j in the Markov chain. This variable is discussed in Chung [17].

5.2.2.2 Two-state Case - Laplace Transform of Wet Period.

In the two-state case, letting $\phi(a,s) = R^{+-}_{12}$, equation (5.42) becomes

$$\frac{\partial}{\partial a}\phi(a,s) + \rho\mu_2^{-1}\phi^2(a,s) + \left[\mu_1^{-1}(\lambda+s) - \mu_2^{-1}(\rho+s)\right]\phi(a,s) - \lambda\mu_1^{-1} = 0 \qquad (5.43)$$

The initial condition in this case comes from the holding time in state μ_1, which is exponential with parameter λ. Hence we have the initial condition

$$\phi(0,s) = \lambda(\lambda+s)^{-1}$$

We will carry out the complete solution for this case.

Equation (5.43) is a Riccati equation, which can be reduced to a second order linear equation by letting

$$\phi(a,s) = y'(a,s)/b_0 y(a,s)$$

where $b_0 = \rho\mu_2^{-1}$

$$b_1 = \mu_1^{-1}(\lambda+s) - \mu_2^{-1}(\rho+s)$$
$$b_2 = -\lambda\mu_1^{-1} \qquad (5.44)$$

The resulting 2^{nd} order linear equation, as can be easily checked, is

$$y''(a,s) + b_1 y'(a,s) + b_0 b_2 y(a,s) = 0 \qquad (5.45)$$

To solve (5.45) we first obtain the characteristic roots:

$$\theta = \frac{\mu_2^{-1}(\rho+s)-\mu_1^{-1}(\lambda+s)}{2} \pm \sqrt{\frac{\left[\mu_2^{-1}(\rho+s)-\mu_1^{-1}(\lambda+s)\right]^2}{4} + \rho\lambda\mu_1^{-1}\mu_2^{-1}} \qquad (5.46)$$

Note that the discriminant is equal to

$$\frac{\mu_2^{-2}(\rho+s)^2}{4} + \frac{\mu_1^{-2}(\lambda+s)^2}{4} - \mu_1^{-1}\mu_2^{-1}s(\rho+\lambda)-\mu_1^{-1}\mu_2^{-1}s^2 \qquad (5.47)$$

which is always positive because $\mu_2 < 0$. Therefore the characteristic roots are real, and the solution for $y(x,s)$ is:

$$y(a,s) = C_1 e^{\theta_1 a} + C_2 e^{\theta_2 a} \qquad (5.48)$$

Substituting (5.48) into (5.44) we obtain

$$\phi(a,s) = \frac{C_1\theta_1 e^{\theta_1 a} + C_2\theta_2 e^{\theta_2 a}}{\rho\mu_2^{-1}\left[C_1 e^{\theta_1 a} + C_2 e^{\theta_2 a}\right]} \qquad (5.49)$$

Now, letting $C = C_2/C_1$, we obtain

$$\phi(a,s) = \frac{\theta_1 e^{\theta_1 a} + C\theta_2 e^{\theta_2 a}}{\rho\mu_2^{-1}\left[e^{\theta_1 a} + Ce^{\theta_2 a}\right]} \qquad (5.50)$$

which involves only one arbitrary constant. To evaluate this constant we use the initial condition to obtain:

$$\phi(0,s) = \frac{\lambda}{\lambda+s} = \frac{\theta_1 + C\,\theta_2}{\rho\mu_2^{-1}(1+c)} \tag{5.51}$$

Hence

$$C = -\frac{(\lambda+s)\theta_1 - \rho\mu_2^{-1}\lambda}{(\lambda+s)\theta_2 - \rho\mu_2^{-1}\lambda} \tag{5.52}$$

So we have as a final result that

$$\phi(a,s) = \frac{\theta_1 e^{\theta_1 a} + C\theta_2 e^{\theta_2 a}}{\rho\mu_2^{-1}\left[e^{\theta_1 a} + Ce^{\theta_2 a}\right]} \tag{5.53}$$

where C is given by (5.52) and θ_1, θ_2 are given by (5.46).

5.3 Moments of Time to First Emptiness for the Two-State Case

Since we have derived the Laplace Transorm of the time to first emptiness, we can use these equations to obtain the moments without having to actually solve for the transform. Although we could in theory write down the moment equations for the general case, the resulting equations are not easily solved, and so we restrict ourselves to looking at the first two moments in the two state case, for which the equations are

$$ET = -\frac{d}{ds}\,\phi_s\Big|_{s=0}$$

$$ET^2 = \frac{d^2}{ds^2}\,\phi_s\Big|_{s=0}$$

so that

$$\mathrm{Var}\,T = \frac{d^2}{ds^2}\,\phi_s\Big|_{s=0} - \left[\frac{d}{ds}\,\phi_s\right]^2\Big|_{s=0}$$

For the semi-infinite dam we differentiate equation (5.23) to obtain

$$2\rho\mu_2^{-1}\phi(s)\phi'(s) - \left[\rho\mu_2^{-1} - \lambda\mu_1^{-1} - s\,(\mu_1^{-1} - \mu_2^{-1})\right]\phi_s' + (\mu_1^{-1} - \mu_2^{-1})\phi_s = 0 \qquad (5.54)$$

And, upon evaluating this expression at $s = 0$ we obtain

$$-2\rho\mu_2^{-1}ET + (\rho\mu_2^{-1} - \lambda\mu_1^{-1})ET + \mu_1^{-1} - \mu_2^{-1} = 0$$

from which we obtain

$$ET = \frac{\mu_2 - \mu_1}{\mu_1\rho + \mu_2\lambda} \qquad (5.55)$$

which can be written in terms of the drift parameter as

$$ET = \left(\frac{1}{m}\right)\left(\frac{\mu_2 - \mu_1}{\rho + \lambda}\right), \quad m \leq 0 \qquad (5.56)$$

Differentiating equation (5.23) again, we obtain

$$2\rho\mu_2^{-1}\left[\phi(s)\phi''(s) + \phi'(s)^2\right] - \left[\rho\mu_2^{-1} - \lambda\mu_1^{-1} - s(\mu_1^{-1} - \mu_2^{-1})\right]\phi''_s + 2[\mu_1^{-1} - \mu_2^{-1}]\phi'_s = 0 \qquad (5.57)$$

Evaluating again at $s = 0$ we obtain

$$2\rho\mu_2^{-1}[ET^2 + E^2T] - [\rho\mu_2^{-1} - \lambda\mu_1^{-1}]ET^2 - 2(\mu_1^{-1} - \mu_2^{-1})ET = 0$$

From which it follows that

$$ET^2 = \frac{2(\mu_1^{-1} - \mu_2^{-1})ET - 2\rho\mu_2^{-1}E^2T}{\rho\mu_2^{-1} - \lambda\mu_2^{-1}} \qquad (5.58)$$

Therefore

$$\text{Var } T = \left(\frac{1}{m}\right)\left[\frac{2(\mu_1^{-1}\ \ \mu_2^{-1})ET - (3\rho\mu_2^{-1}-\lambda\mu_1^{-1})E^2T}{\rho + \lambda}\right] \tag{5.59}$$

Similar calculations for the finite dam produce

$$ET = (\lambda\mu_1^{-1} + \rho\mu_2^{-1})\left[\mu_1^{-1}-\mu_2^{-1}+\lambda^{-1}\mu_2^{-1}(\lambda+\rho)e^{-(\lambda\mu_1^{-1}+\rho\mu_2^{-1})a}\right] \tag{5.60}$$

5.3.1 Conditions for Recurrence

In analogy with the theory of Markov chains, we call the contents process C_t recurrent or transient depending on whether $p = 1$ or $p < 1$ where $p = P[T < \infty]$. In the recurrent case we call C_t positive recurrent or null recurrent depending on whether $ET < \infty$ or $ET = \infty$. We will relate conditions for recurrence to the drift parameter of the process, m .

For the topless dam, it follows from equation (5.27) that $m \leq 0$ implies recurrence, while $m > 0$ implies transience. From equation (5.56) it follows that $m=0$ implies null recurrence while $m < 0$ implies positive recurrence.

By symmetry, for the bottomless dam it follows that $m > 0$ implies positive recurrence, $m=0$ implies null recurrence, and $m < 0$ implies transience.

5.4 Limiting Probability of Emptiness for the Two-State Case

Consider the time-dependent probability of emptiness

$$P_e(t) = P(C_t = 0) .$$

If the bivariate process has a limiting distribution, then we consider

the following limit

$$P_e = \lim_{t\to\infty} P_e(t)$$

We call this the limiting probability of emptiness, and it can be interpreted as the large-time percentage of time that the dam will be found empty.

Brockwell [12] has actually derived the limiting distribution of the content by solving the Kolmogorov equations. However, as is the case with the time to first emptiness, these equations require knowledge of the boundary value P_e for their solution. Although P_e can be obtained by other methods, we present here a direct method of evaluating P_e by using the previous results concerning the time to first emptiness and a renewal argument.

5.4.1 Embedded Regenerative Process

Assuming that $X(0) = \mu_1 > 0$, define

$$\tau_1 = \inf\{t > 0 | C_t = 0\}$$

$$\tau_2 = \inf\{t > \tau_1 | C_t = 0\}$$

and, continuing in this fashion, define τ_i , $i \geq 1$. The τ_i's are the first return times to zero of the contents process, and since the bivariate process (X_t, C_t) is strong Markov and the τ's are stopping times, then they are regeneration points and the process $(\tau_1, \tau_1 + \tau_2, \ldots)$ is an embedded delayed renewal process (see Figure 5.1).

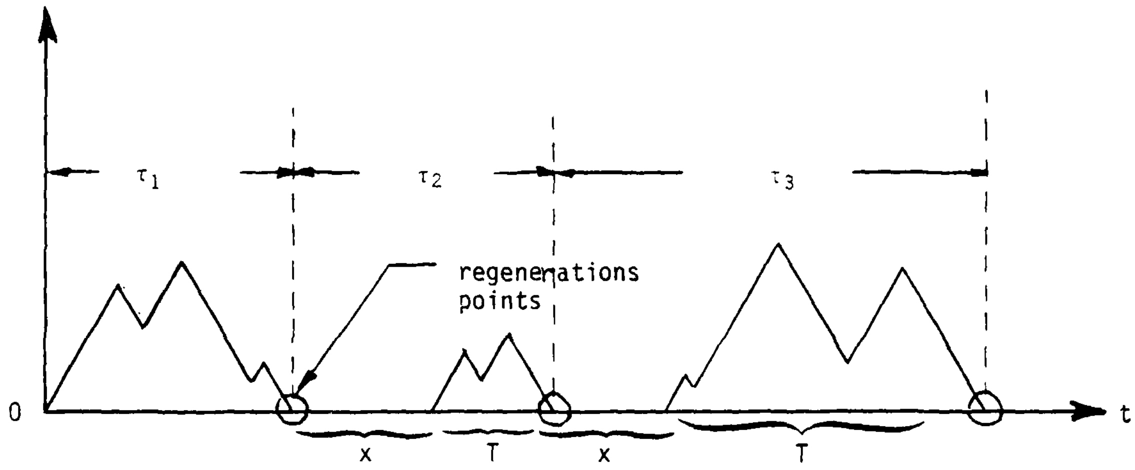


Figure 5.1 Embedded Renewal Process.

Now, thinking of this as a two-cycle renewal process with cycles T and X, then it is a standard result of the renewal theorem that

$$P_e = \frac{EX}{EX + ET} \tag{5.61}$$

5.4.2 Semi-Infinite Topless Dam

For the semi-infinite topless dam we have

$$ET = \left(\frac{1}{m}\right)\left(\frac{\mu_2-\mu_1}{\rho+\lambda}\right) \tag{5.62}$$

and, by the memoryless property of the exponential,

$$X \sim \exp(\rho) \tag{5.63}$$

so that $EX = \rho^{-1}$.

Substituting (5.62) and (5.63) into (5.61) we obtain after some algebra

$$P_e = \frac{m}{\mu_2} \tag{5.64}$$

5.4.3 Semi-Infinite Bottomless Dam - Limiting Probability of Overflow

Using the symmetry between the topless and the bottomless version, we can write for the time to first overflow, T

$$ET = \left(\frac{1}{m}\right)\left(\frac{\mu_1-\mu_2}{\lambda+\rho}\right) \tag{5.65}$$

and, using the analogous embedded renewal process, we have for the limiting probability of overflow, P_f ,

$$P_f = \frac{\lambda^{-1}}{\lambda^{-1}+\dfrac{\mu_1-\mu_2}{\mu_2\lambda+\mu_1\rho}} = \frac{m}{\mu_1} \tag{5.66}$$

5.4.4 Finite Dam

In this case, substituting equations (5.60) and (5.63) into (5.61) we obtain

$$P_e = \lambda[\lambda\mu_1^{-1} + \rho\mu_2^{-1}]\left[(\lambda+\rho)(\lambda\mu_1^{-1} + \rho\mu_2^{-1}e^{-(\lambda\mu_1^{-1} + \rho\mu_2^{-1})a}\right]^{-1} \tag{5.67}$$

For the case $\mu_1 = 1$, $\mu_2 = -1$ we obtain

$$P_e = \lambda(\rho-\lambda)\left[(\lambda+\rho)\left(\rho e^{(\rho-\lambda)a} - \lambda\right)\right]^{-1} \tag{5.68}$$

which agress with the previously reported result [12].

CHAPTER VI

A FUNCTIONAL CENTRAL LIMIT THEOREM FOR THE UNRESTRICTED CONTENTS PROCESS ON TWO STATES

6.1 Formulation

In this chapter we show that a scaled version of the contents process C_t, $0 \leq t < \infty$, will converge weakly to the Wiener process on $D[0,\infty)$. A short description of the topology of this space was provided in Chapter II. The weak convergence results of this section generalize the results of Fukushima and Hitsuda [22] and Pinsky [38], who showed convergence of the marginal distributions.

To establish weak convergence for C_t we appeal to a result given in Chapter II that if T_i are the times of jumps of a Markov chain and $\{X_n\}$ are the succession of states visited, then if the sequence $\{X_n\}$ is deterministic, the holding times are unconditionally independent, i.e.,

$$P[T_j - T_{j-1} > u_j, j=1,\ldots,m] = \prod_{j=1}^{n} e^{-\lambda_{j-1} u_j} \quad .$$

In the two-state case the sequence X_n is deterministic, being given by $\{\mu_1,\mu_2,\mu_1,\mu_2,\ldots\}$. If we fix on the entrance times into the μ_1 state by the X_t process, we see that we can consider an embedded renewal process with cycle lengths (ξ_i) where

$$\xi_0 = 0$$

$$\xi_i = T_{2i} - T_{2(i-1)}$$

as seen in Figure 6.1.

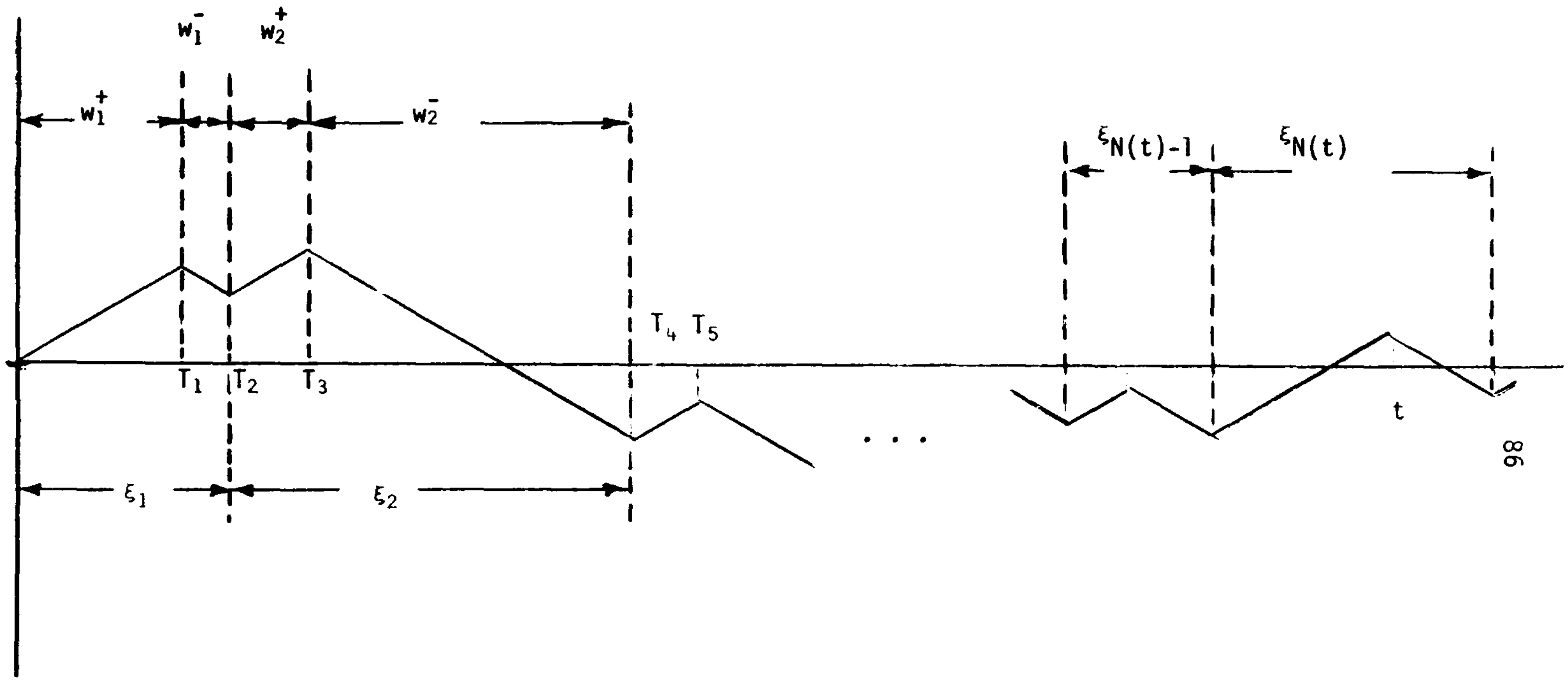


Figure 6.1 Embedded Renewal Process

From Figure 6.1 we can see that the embedded renewal process is really an alternating renewal perocess with two subcycles in each interval. Thus the first cycle has length ξ_1, and is composed of the two subcylces of length $T_1 \sim \exp(\lambda)$ and $T_2 - T_1 \sim \exp(\rho)$. The second cycle has length ξ_2 and is composed of the two subcycles of length $T_3 - T_2 \sim \exp(\lambda)$ and $T_4 - T_3 \sim \exp(\rho)$, and so forth. The slopes of the sample path over odd subcycles is $\mu_1(>0)$ and over even subcycles is $\mu_2(<0)$. We use the notation W_i^+ to represent the waiting time random variable for the subcycle of the i^{th} cycle during which the slope of the sample path is positive, and W_i^- for the subcycle during which it is negative. Thus

$$W_i^+ = T_{2i-1} - T_{2(i-1)}$$

and

$$W_i^- = T_{2i} - T_{2i-1} \quad .$$

Note that $W_i^+ + W_i^- = T_{2i} - T_{2(i-1)} = \xi_i$. Let $\{S_n\}_{n=1}^{\infty}$ be the renewal process, so that

$$S_n = \sum_{i=1}^{n} \xi_i \tag{6.1}$$

Let $N(t)$ be the renewal counting function for the S_n process; thus

$$N(t) = \inf \{j | \sum_{i=1}^{j} \xi_i \geq t\} \tag{6.2}$$

Now consider

$$C(nt) = \int_0^{nt} x(s)ds \tag{6.3}$$

Then, referring to Figure 6.1, we can write $C(nt)$ as

$$C(nt) = \int_0^{S_{N(nt)}} x(u)du - \int_{nt}^{S_{N(nt)}} x(u)du \tag{6.4}$$

where the second term represents the 'overshoot' of the process to the first renewal past nt, and so must be subtracted off to maintain equality.

Since the sample paths of $X(t)$ are piecewise linear with slopes μ_1 and μ_2, we can re-write $C(nt)$ in terms of the holding time variables $W_i^{\pm}$ as

$$C(nt) = \sum_i^{N(nt)} W_i^+ \mu_1 + \sum_i^{N(nt)} W_i^- \mu_2 - R_n(t) \tag{6.5}$$

where

$$R_n(t) = \int_{nt}^{S_{N(nt)}} x(u)du \tag{6.6}$$

Now define

$$Z_n = \sum_i^n \left[W_i^+ \mu_1 + W_i^- \mu_2 - \frac{\mu_1}{\lambda} - \frac{\mu_2}{\rho} \right] \tag{6.7}$$

Then, by Donsker's theorem (Billingsley [9]),

$$\frac{Z_{[n\cdot]}}{\sqrt{n}} \Rightarrow W\left(\left(\frac{\mu_1^2}{\lambda^2} + \frac{\mu_2^2}{\rho^2}\right) \cdot\right) \quad \text{on} \quad D[0,\infty) \tag{6.8}$$

Returning to the original process (6.5), we can write

$$C(nt) = \sum_1^{N(nt)} \left(W_i^+ \mu_1 + W_i^- \mu_2 - \frac{\mu_1}{\lambda} - \frac{\mu_2}{\rho} \right)$$

$$+ \left(\frac{\mu_1}{\lambda} + \frac{\mu_2}{\rho} \right) N(nt) + R_n(t) \tag{6.9}$$

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or,

$$C(nt) = Z_{N(nt)} + \left(\frac{\mu_1}{\lambda} + \frac{\mu_2}{\rho}\right) N(nt) + R_n(t) \tag{6.10}$$

Setting $\eta_n(t) = \frac{N(nt)}{n}$, then $\{\eta_n(t)\}_{n=1}^{\infty}$ are increasing processes in t , and for each fixed t ,

$$\eta_n(t) \xrightarrow{P} \frac{t}{E\xi} \quad \text{as} \quad n\to\infty \ .$$

Therefore it follows that

$$\eta_n(\cdot) \Rightarrow [E(\xi)]^{-1}(\cdot) \quad \text{on} \quad D[0,\infty),\ \text{i.e.,}$$

$$\frac{N(n\cdot)}{n} \Rightarrow \left(\frac{1}{\lambda} + \frac{1}{\rho}\right)^{-1}(\cdot) \tag{6.11}$$

Since the limit is degenerate, then the following joint convergence holds by Theorem 4.4 in Billingsley [9].

$$\left(\frac{Z_{[n\cdot]}}{\sqrt{n}}\ ,\ \frac{N(n\cdot)}{n}\right) \Rightarrow \left(W\left(\left[\frac{\mu_1^2}{\lambda^2} + \frac{\mu_2^2}{\rho^2}\right] \quad (\cdot)\right)\ ,\ \left(\frac{1}{\lambda} + \frac{1}{\rho}\right)^{-1}(\cdot)\right) \tag{6.12}$$

Using the continuous mapping

$$\psi(x,\phi) = x \cdot \phi$$

we conclude that

$$\frac{Z_{[N(n\cdot)]}}{\sqrt{n}} \Rightarrow W(\beta\cdot) \quad \text{on} \quad D[0,\infty) \tag{6.13}$$

when $\quad \beta = \left(\frac{\mu_1^2}{\lambda^2} + \frac{\mu_2^2}{\rho^2}\right) \left(\frac{1}{\lambda} + \frac{1}{\rho}\right)^{-1}$

We now show that $\frac{R_n}{\sqrt{n}} \Rightarrow 0$ in $D[0,\infty)$. From 6.6 we have

$$R_n(t) = \int_{nt}^{S_{N(nt)}} x(u)du$$

$$\leq \int_{nt}^{S_{N(nt)}} \sup_{nt \leq u \leq S_{N(nt)}} x(u)\, du$$

$$\leq V(\mu_1,-\mu_2)\,(S_{N(nt)} - nt)$$

$$= V(\mu_1,-\mu_2)\,\gamma(nt)$$

where $\gamma(nt)$ is the forward recurrence time evaluated at nt. Therefore it suffices to show that

$$\frac{\gamma(n\cdot)}{\sqrt{n}} \Rightarrow 0 \quad \text{in} \quad D[0,\infty) .$$

If μ,σ^2 are the mean and variance of the interarrival times for a renewal process, then Iglehart and Whitt [26.1] have shown that

$$\left(\frac{S_{[n\cdot]} - \mu n\cdot}{\sigma\sqrt{n}} , \frac{N(n\cdot) - n\cdot/\mu}{\sqrt{n\sigma^2/\mu^3}}\right) \Rightarrow \Big(W(\cdot), -W(\cdot)\Big) . \tag{6.14}$$

Now, since $\frac{N(n\cdot)}{n} \Rightarrow \frac{(\cdot)}{\mu}$ in $D[0,\infty)$, then we also have the joint convergence

$$\left(\frac{S_{[n\cdot]} - \mu n\cdot}{\sigma\sqrt{n}} , \frac{\mu N(n\cdot) - n\cdot}{\sigma\sqrt{n}} , \frac{N(n\cdot)}{n}\right) \Rightarrow \left(W(\cdot), - W\left(\frac{\cdot}{\mu}\right), \frac{\cdot}{\mu}\right) \tag{6.15}$$

Now consider the mapping $\psi(x,y,\phi) = (x\circ\phi,y)$ and apply the continuous mapping theorem to obtain

$$\left(\frac{S_{N(n\cdot)} - \mu N(n\cdot)}{\sigma\sqrt{n}}, \frac{\mu N(n\cdot) - n\cdot}{\sigma\sqrt{n}}\right) \Rightarrow \left(W\left(\frac{\cdot}{\mu}\right), -W\left(\frac{\cdot}{\mu}\right)\right) \tag{6.16}$$

and, again applying the continuous mapping theorem to

$$\tilde{\psi}(x,y) = x + y$$

we obtain

$$\frac{S_{N(n\cdot)} - \mu N(n\cdot) + \mu N(n\cdot) - n\cdot}{\sigma\sqrt{n}} \Rightarrow W\left(\frac{\cdot}{\mu}\right) + \left(-W\left(\frac{\cdot}{\mu}\right)\right)$$

or $$\frac{S_{N(n\cdot)} - n\cdot}{\sigma\sqrt{n}} \Rightarrow 0$$

so that indeed $\frac{\gamma(n\cdot)}{\sqrt{n}} \Rightarrow 0$ on $D[0,\infty)$.

6.2 Zero Drift

In the zero drift case $\frac{\mu_1}{\lambda} + \frac{\mu_2}{\rho} = 0$, so that from 6.10 it follows that

$$C(nt) = Z_{N(nt)} + R_n(t) \tag{6.17}$$

and, using Th. 4.1 in Billingsley [9] ,

$$\frac{C(n\cdot)}{\sqrt{n}} \Rightarrow W(\beta\cdot) \quad \text{on} \quad D[0,\infty) .$$

Hence, if $\Lambda \varepsilon B$, the sigma field generated by the open sets relative to the $D[0,\infty)$ metric,

$$\left| P\left[\frac{C(n\cdot)}{\sqrt{n}} \varepsilon \Lambda\right] - P\left[W(\beta\cdot) \varepsilon \Lambda\right] \right| \to 0 \quad \text{as} \quad n \to \infty$$

Thus we have the large-n approximation

$$P\left[\frac{C(n\cdot)}{\sqrt{n}} \varepsilon \Lambda\right] \approx P\left[W(\beta\cdot) \varepsilon \Lambda\right] .$$

6.3 Non-Zero Drift

For non-zero drift Eq. (6.10) is

$$C(nt) = Z_{N(nt)} + \left(\frac{\mu_1}{\lambda} + \frac{\mu_2}{\rho}\right) N(nt) + R_n(t)$$

so that if

$$C^*(nt) = \frac{C(nt)}{\sqrt{n}} - \left(\frac{\mu_1}{\lambda} + \frac{\mu_2}{\rho}\right) \frac{N(nt)}{\sqrt{n}}$$

then

$$C^*(n\cdot) \Rightarrow W(\beta\cdot) \quad \text{on} \quad D[0,\infty) .$$

Thus we see that in this case the centering function is random.

6.4 Asymptotic Behavior of Range

We can obtain the asymptotic distribution of the range from the corresponding result for Brownian motion as follows: Consider the continuous mapping from $D[0,\infty)$ into R given by

$$f\big(C(nt)\big) = \left(\underset{0\le u\le t}{\text{Max}} \; C(nu) - \underset{0\le u\le t}{\text{Min}} \; C(nu)\right) .$$

Then by the continuous mapping theorem, for $R(nt)$, the range of $\frac{C(nt)}{\sqrt{n}}$, we have

$$R(nt) \Rightarrow R^B(\beta t) ,$$

the range of Browian motion.

W. Feller [21] derived the density of the range of Brownian motion as:

$$f(t,r) = 8 \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \phi\left(\frac{kr}{t^{1/2}}\right)$$

where $\phi(x)$ is the standard normal density function. Hence the asymptotic distribution of $R(t)$ has a density given by

$$8 \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \phi\left(\frac{kr}{\beta^{\frac{1}{2}} t^{\frac{1}{2}}}\right) .$$

CHAPTER VII

SUMMARY AND CONCLUSIONS

7.1 Summary

In this investigation we considered several problems connected with Markovian storage models in continuous time. We proceeded by considering three different variations of the model. These were the doubly-infinite dam without top or bottom, the semi-infinite dam in both its topless and bottomless versions, and the finite dam with both a top and a bottom. In each case the process which we examined was the bivariate Markov process (X_t, C_t) in which X_t was a Markov chain with finite state space representing the net input rate, and C_t was the integral of X_t, and represented the dam contents. We maintained as much generality as possible throughout, but restricted overselves to particular cases whenever necessary. The most general formulation of the model was when the Markov chain X_t was defined an an arbitrary n-dimensional state space with an arbitrary generator matrix. We often restricted the discussion to the two-state case in which the Markov chain was defined on states $\mu_1 > 0$ and $\mu_2 < 0$ and the holding times in the states were exponential with parameters λ and ρ. On occasion we restricted ourselves still further to the symmetric case in which $\lambda = \rho$ and $\mu_1 = -\mu_2$.

In the derivations, we did not follow the formal definition-theorem style of presentation, but rather chose to present the derivations and results in a more natural flow. Since with this style of presentation it is not as easy to locate individual results, we now present the specific results derived in each chapter, as an aid to the reader.

7.2 Conclusions

Chapter III

In this chapter we derived several moment functions for the Markov chain process X_t and the contents process C_t.

(i) General Case - Markov chain:

$$EX_t = \sum_i \mu_i \pi_i(t)$$

$$EX_t X_{t+s} = \sum_i \sum_j \mu_i \mu_j P_{ij}(s) \pi_i(t)$$

$$\text{Var } X_t = \sum_{i \neq} \sum_j \mu_i(\mu_i - \mu_j) \pi_i(t) \pi_j(t)$$

$$\text{Cov}[X_t, X_{t+s}] = \sum_i \sum_j \mu_i \mu_j P_{ij}(s) \pi_i(t) - \sum_i \mu_i \pi_i(t) \sum_j \mu_i \pi_i(t+s)$$

When X_t is stationary,

$$\text{Corr}[X_t, X_{t+s}] = \sum_{k=2}^{m} C_k e^{\theta_k s}$$

where

$$C_k = \frac{\sum_i \sum_j t_{ki}\ r_{ki} \pi_i \mu_i \mu_j}{\sum_{i \neq j} \sum \mu_i(\mu_i - \mu_j) \pi_i \pi_j}$$

and $\underset{\sim}{t}_k$, $\underset{\sim}{r}_k$, $k = 2, \ldots, m$ are the normalized right and left eigen-vectors of the generator matrix, with eigenvalues θ_k. The asymptotic behavior of ρ_s is governed by the largest non-zero eigenvalue.

(ii) General Case - Contents process:

$$EC_t = \sum \mu_i \pi_i t .$$

If $\Delta_h(t) = \int_{t-\frac{h}{2}}^{t+\frac{h}{2}} X(u)du$, an increment of the C_t process of length h centered at t, then

$$E\left(\Delta_h^2(t)\right) = 2\sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki}r_{kj}}{\theta_k^2}\left[e^{\theta_k h} - \theta_k h - 1\right] \mu_i\mu_j\pi_i$$

$$+ \sum_i \sum_j \mu_i\mu_j\pi_i\pi_j\, h^2$$

$$\mathrm{Var}\left(\Delta_h(t)\right) = 2\sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki}r_{kj}}{\theta_k^2}\left[e^{\theta_k h} - \theta_k h - 1\right]\mu_i\mu_j\pi_i$$

$$\mathrm{Cov}\left[\Delta_h(t),\Delta h(t+h)\right] = \sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki}r_{kj}}{\theta_k^2}\left[1 + e^{2\theta_k h} - 2e^{\theta_k h}\right]\mu_i\mu_j\pi_i$$

$$\mathrm{Corr}\left[\Delta_h(t),\Delta_h(t+h)\right] = \frac{\sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki}r_{kj}}{\theta_k^2}\left[1+e^{2\theta_k h}-2e^{\theta_k h}\right]\mu_i\mu_j\pi_i}{2\sum_i \sum_j \sum_{k=2}^{m} \frac{t_{ki}r_{kj}}{\theta_k^2}\left[e^{\theta_k h}-\theta_k h-1\right]\mu_i\mu_j\pi_i}$$

(iii) <u>Two-State Case - Markov chain</u>

The moment functions are:

$$EX_t = (\lambda+\rho)^{-1}(\rho\mu_1+\lambda\mu_2)$$

$$EX_t^2 = (\lambda+\rho)^{-1}(\rho\mu_1^2+\lambda\mu_2^2)$$

$$\mathrm{Var}\, X_t = \rho\lambda(\lambda+\rho)^{-2}(\mu_1-\mu_2)^2$$

$$EX_tX_{t+s} = (\lambda+\rho)^{-2}\left[(\rho\mu_1+\lambda\mu_2)^2 + \lambda\rho(\mu_1-\mu_2)^2 e^{-(\lambda+\rho)s}\right]$$

$$\mathrm{Cov}[X_t, X_{t+s}] = \rho\lambda(\lambda+\rho)^{-2}(\mu_1-\mu_2)^2 e^{-(\lambda+\rho)s}$$

$$\mathrm{Corr}[X_t, X_{t+s}] = e^{-(\lambda+\rho)s}$$

(iv) Two-State Case - Contents process

$$EC_t = (\rho+\lambda)^{-1}(\rho\mu_1+\lambda\mu_2)t$$

$$\mathrm{Var}\ C_t = 2(\rho+\lambda)^{-4}\rho\lambda(\mu_1-\mu_2)^2\left[e^{-(\lambda+\rho)t} + (\lambda+\rho)t-1\right]$$

$$\mathrm{Corr}[\Delta_h(t), \Delta_h(t+h)] = \frac{\left[1-e^{-(\lambda+\rho)h}\right]^2}{2\left[(\lambda+\rho)h-1+e^{-(\lambda+\rho)h}\right]}$$

In this case the correlation function is free of the states.

Chapter IV

In this chapter we studied the range of the contents process for the doubly infinite dam.

(i) General Case

We derived the Laplace transform with respect to time of a form of the joint distribution of the maximum and minimum variables.

If $M_s = \mathrm{Sup}\{C_u,\ 0 \le u \le T,\ T \sim \exp(s)\}$

$m_s = \mathrm{Inf}\{C_u,\ 0 \le u \le T,\ T \sim \exp(s)\}$

and

$$\psi_i(x,s) = 1-P[M_s \le a - x,\ m_s \ge -x \mid X_0 = \mu_i]$$

then

$$\underset{\sim}{\psi}(x,s) = \exp[-D^{-1}(Q-sI)x]\ \underset{\sim}{\psi}(o,s)$$

where

$$\psi_i(a) = 1, \ \mu_i > 0$$

$$\psi_i(0) = 1, \ \mu_i < 0 \ .$$

(ii) Two-State Case

The distribution of M_s is given by

$$P\left[M_s > y|\mu_i\right] = \begin{cases} e^{-\theta y}, & \mu_i > 0 \\ \dfrac{\lambda + s - \theta\mu_1}{\lambda} e^{-\theta y}, & \mu_i < 0 \end{cases}$$

where

$$\theta = \frac{1}{2} A + \left[\frac{1}{4} A^2 - \frac{s(\lambda+\rho)+s^2}{\mu_1\mu_2}\right]$$

where

$$A = \frac{\rho + s}{\mu_2} + \frac{\lambda + s}{\mu_1} \quad .$$

Hence if the initial rate is positive, the maximum to time T is exponentially distributed. If the initial rate is negative, the maximum has a truncated exponential distribution with a mass at 0 of size $\frac{\theta_1\mu_1 - s}{\lambda}$.

If $R_s = M_s - m_s$, the range to time T , we found that when X_t is stationary,

$$ER_s = (\lambda+\rho)^{-1}\Big\{\theta_1^{-1}\,(\rho+\lambda+s-\theta_1\mu_1)$$

$$+ \tilde{\theta}_1^{-1}[\lambda+\rho\lambda^{-1}(\lambda+s+\tilde{\theta}_1\mu_2)]\Big\}$$

where $\theta_1 = \theta$ above and

$\tilde{\theta}_1 = \theta - A$.

(iii) Symmetric Case

For the symmetric case in which $\mu_1 = b$, $\mu_2 = -b$, and $\lambda = \rho = a$, we found for the range to time t, say R_t , that

$$ER_t = \frac{b}{a}\left[F\left(-\frac{1}{2}, 1, -2at\right) - 1\right] = \frac{b}{a}\left[e^{-2at} F\left(\frac{3}{2}, 1, 2at\right) - 1\right]$$

where $F(\alpha,\beta,x)$ is the confluent hypergeometric series defined by

$$F(\alpha,\beta,x) = \sum_{j=0}^{\infty} \frac{(\alpha)_j}{(\gamma)_j} \frac{x^j}{j!}$$

where $$(\alpha)_j = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+j-1) & \text{for } j=1,2,\ldots \\ 1 & \text{for } j=0 \end{cases}$$

Using an asymptotic expansion of the Laplace transform we found the asymptotic behavior of the range as

$$m_R(t) - \frac{b}{\sqrt{a}}\left[\sqrt{\frac{8}{\pi}}\, t^{\frac{1}{2}} - \frac{1}{\sqrt{a}}\right] - \frac{3}{4}\sqrt{\frac{2}{\pi}}\, \frac{b}{a^{3/2}}\, t^{-\frac{1}{2}} = 0\left(\frac{\sqrt{2}}{32}\, \frac{b}{a^{5/2}}\, \frac{t^{-3/2}}{\Gamma(-1/2)}\right)$$

so that in particular a large-t value is

$$m_R(t) \sim \frac{b}{\sqrt{a}}\left[\sqrt{\frac{8}{\pi}}\, t^{\frac{1}{2}}\right] .$$

We did an exact calculation of ER_t for $0 \leq t \leq 10$ in the case $a = b = 1$ and found that the Hurst phenomenon was evident in the approximate range $0 \leq t \leq 2$. If a, b are chosen so that $b = a^{\frac{1}{2}}$, the asymptotic behavior is not changed. In this case, choosing a to be small extends the Hurst behavior to longer periods of time, while at the same time it increases the correlation between increments.

In order to explain the Hurst phenomenon with this model for Hurst's longest series of about 2000 years requires a correlation between adjacent yearly increments of .99865, which is unrealistically high. For the shorter time period of 20 years, the required correlation drops to a more acceptable value of .87720.

Chapter V

In this chapter we introduced the principles of invariance developed by Chandrasekhar and Bellman in physical models to the study of storage models. The relevance of this chapter is in the power of these techniques to provide direct solutions to first passage problems in a simplified manner over classical techniques.

(i) Semi-Infinite Topless Dam

General Case

We derived Laplace transform equations for the distribution of the first return to zero, given an exit rate $\mu_i > 0$ and return rate $\mu_j < 0$.

Two-State Case

In this case the first return time to zero corresponds to the wet period. We solved the general system of Laplace transform equations explicitly to obtain the Laplace transform of the wet period, T, as

$$\phi_s = \mu_2(2\rho)^{-1}[A + (A^2 - 4\lambda\rho\mu_1^{-1}\mu_2^{-1})^{\frac{1}{2}}]$$

where $\quad A = s(\mu_2^{-1}-\mu_1^{-1}) + \rho\mu_2^{-1} - \lambda\mu_1^{-1}$

from which we obtained

$$P[T<\infty] = \begin{cases} i & \text{if } m \leq 0 \\ -\dfrac{\rho\mu_1}{\lambda\mu_2} & \text{if } m = 0 \end{cases}$$

By differentiating ϕ_s, we obtained (for $m \leq 0$).

$$ET = \frac{1}{m} \quad \frac{\mu_2-\mu_1}{\rho + \lambda}$$

$$\text{Var } T = \frac{2(\mu_1{}^{-1}\mu_2{}^{-1})ET - (3\rho\mu_2{}^{-1}-\lambda\mu_1{}^{-1})E^2T}{m(\rho+\lambda)}$$

where m is the drift, i.e., $m = EX_t$ for X_t stationary.

We then used a renewal argument to obtain the limiting probability of emptiness, P_e as

$$P_e = \frac{m}{\mu_2} \quad (\text{for } m \leq 0)$$

Symmetric Case

In the symmetric case we were able to invert the Laplace transform to obtain the density of the wet period as

$$f_T(x) = \frac{a}{2} F\left(\frac{3}{2}, 3, -2ax\right)$$

where F is again the confluent hypergeometric function.

(ii) Semi-Infinite Bottomless Dam

In this case we derived the Laplace transform of the time between overflows as

$$\phi_s = \mu_1(2\lambda)^{-1}\left[A - (A^2+ 4\lambda\rho\mu_1{}^{-1}\mu_2{}^{-1})^{1/2}\right]$$

where $\quad A = s(\mu_1{}^{-1} - \mu_2{}^{-1}) + \lambda\mu_1{}^{-1} - \rho\mu_2{}^{-1}$

The density of the time between overflows is the same as the density of the wet period for the topless dam.

We derived the limiting probability of overflow, P_f , as

$$P_f = m/\mu_1 \ .$$

(iii) Conditions for Recurrence

We derived necessary and sufficient conditions for recurrence of the C_t process as follows:

Topless Dam: C_t is recurrent iff $m = EX_t \le 0$. It is positive recurrent if $m < 0$ and null recurrent if $m = 0$.

Bottomless Dam: C_t is recurrent iff $m \ge 0$. It is positive recurrent if $m > 0$ and null recurrent if $m = 0$.

(iv) Finite Dam

General Case

We derived an initial-value differential equation for the Laplace transform of the wet period.

Two-State Case

We derived explicitly the Laplace transform of the wet period for a dam of height a as

$$\phi(a,s) = \frac{\theta_1 e^{\theta_1 a} + C\theta_2 e^{\theta_2 a}}{\rho\mu_2^{-1}\left[e^{\theta_1 a} + Ce^{\theta_2 a}\right]}$$

where

$$C = -\frac{(\lambda+s)\theta_1 - \rho\mu_2^{-1}\lambda}{(\lambda+s)\theta_2 - \rho\mu_2^{-1}\lambda},$$

$$\theta_{1,2} = A \pm [A^2 + \rho\lambda\mu_1^{-1}\mu_2^{-1}] \text{ and}$$

$$A = \frac{\mu_2^{-1}(\rho+s) - \mu_1^{-1}(\lambda+s)}{2}$$

and we obtained the moment of T as

$$ET = (\lambda\mu_1^{-1}+\rho\mu_2^{-1})^{-1}\left[\mu_1^{-1}-\mu_2^{-1}+\lambda^{-1}\mu_2^{-1}(\lambda+\rho)e^{-(\lambda\mu_1^{-1}+\rho\mu_2^{-1})a}\right]$$

and the limiting probability of overflow as

$$P_e = \lambda(\rho-\lambda)\left[(\lambda+\rho)(\rho e^{(\rho-\lambda)a}-\lambda)\right]^{-1}$$

Chapter VI

In this chapter we established weak convergence in $D[0,\infty)$ of the unrestricted contents process C_t on two states. The basic results which we obtained involved weak convergence to Brownian motion of the process of $C(nt)$. We established that for the zero drift case, $\frac{C(n\cdot)}{\sqrt{n}} \to w(\beta\cdot)$ on $D[0,\infty)$, where

$$\beta = \frac{\rho^2\mu_1^2 + \lambda^2\mu_2^2}{\lambda\rho(\lambda+\rho)}$$

and $W(\cdot)$ represents Wiener measure.

Using this we were able to conclude that the asymptotic distribution of the range has density

$$f_R(t,r) = 8 \sum_{k=1}^{\infty} (-1)^{k-1}k^2 \left(\frac{kr}{\beta^{1/2}t^{1/2}}\right)$$

7.3 Recommendations for Further Study

One possible line of further research would be the extension of the results derived for the two-state case to three or more states. This would require a clever choice of a generator matrix which would give sufficient generality and yet have enough structure to make the calculations manageable. One possibility which we would recommend for further analyses is the generator matrix

$$Q = \rho(\underset{\sim}{1}\underset{\sim}{\pi}' - I) , \quad \rho > 0$$

which has the property that $\underset{\sim}{\pi}$ is the stationary distribution and the autocorrelation function is

$$\mathrm{Corr}[X_t, X_{t+s}] = e^{-\rho s} .$$

Another possible line of research is the extension of the technique of invariance to obtain further results in storage theory. These techniques have been highly developed in the physical fields. What we have presented here is but the briefest introduction to their use in storage theory. Undoubtedly many other results must be obtainable through sufficiently clever applications of these principles.

BIBLIOGRAPHY

1. Ali Khan, M. and Gani, J. "Infinite dams with inputs forming a Markov chain." J. Appl. Prob. 5, 72-83 (1968).

2. Ali Khan, M. "Finite dams with inputs forming a Markov chain." J. Appl. Prob. 7, 291-303 (1970).

3. Ambarzumian, V. "Diffuse reflection of light by a foggy medium." C. R. Acad. Sci. U.R.S.S. 38(8), 229 (1943).

4. Bellman, R., Kalaba, R., and Wing, G. "Invariant imbedding and neutron transport theory - II: functional equations." J. Math. Mech. 7, 741-756 (1958).

5. Bellman, R., and Kalaba, R. "On the principle of invariant imbedding and propagation through inhomogeneous media." Proc. Nat. Acad. Sci. U.S.A. 42, 629-632 (1956).

6. Bellman, R., Kalaba, R., and Prestrud, M. Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness. Elsevier Publishing Co., New York (1963).

7. Bellman, R., Kalaba, R., Prestrud, M., and Kagiwada, H. Invariant Imbedding and Time-Dependent Transport Processes. Elsevier Publishing Co., New York (1964).

8. Bhattacharya, P. and Brockwell, P. "The minimum of an additive process with applications to signal estimation and storage theory," Z. Wahrscheinlichkeitstheorie verw. Gebiete 37, 51-75 (1976).

9. Billingsley, P. Convergence of Probability Measures. Wiley and Sons, Inc., New York (1968).

10. Boes, D. and Salas-La Cruz, J. "On the expected range and expected adjusted range of partial sums of exchangeable random variables." J. Appl. Prob. 10, 671-677 (1973).

11. Brockwell, P., and Gani, J. "A population process with Markovian progenies." J. Math. Anal. Appl. 32, 264-273 (1970).

12. Brockwell, P. "A storage model in which the net growth-rate is a Markov chain." J. Appl. Prob. 9, 129-139 (1972).

13. Brockwell, P. "On the spectrum of a class of matrices arising in storage theory." Z. Wahrscheinlichkeitstheorie verw. Gebrete 25, 253-260 (1973).

14. Brockwell, P. "The minimum of an additive process with applications to signal estimation and storage theory." Z. Wahrscheinlichkeitstheorie verw. Gebiete 37, 51-75 (1976).

15. Buchholz, H. The Confluent Hypergeometric Function. Springer Tracts in Nat. Phil. Vol. 15, Springer-Verlag, New York (1969).

16. Chandrasekhar, S. Radiative Transfer, 2nd ed. Dover Publications, Inc. New York (1960).

17. Chung, K. Markov Chains with Stationary Transition Probabilities, 2nd ed. Springer-Verlag, Berlin (1967).

18. Cinlar, E. Introduction to Stochastic Processes. Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1975).

19. Doetsch, G. Guide to the Applications of Laplace Transforms. Van Nostrand Company, Ltd., London (1961).

20. Durret, R. and Resnick, S. "Weak convergence with random indices." J. Stoch. Pro. & Appl. 5, 213-220 (1976).

20-1 Erickson, R. V. "Stationary measures for a class of storage models driven by a Markov chain." Ann. Math. Statistics 43, 997-1007 (1972).

21 Feller, W. "The asymptotic distribution of the range of sums of independent random variables." Annals of Math. Statistics 22, 427-432 (1951).

22. Fukushima, M., and Hitsuda, M. "On a class of Markov processes taking values on lines and the central limit theorem." Nagoya Math. J. 30, 47-56 (1967).

23. Gani, J. "Some problems on the theory of provisioning and of dams." Biometrika 42, 179-200 (1955).

24. Gomide, F. "Range and deficit analysis using Markov chains." Ph.D. Dissertation, Colorado State University, Fort Collins, CO., 1975.

25. Hille, E. Lectures on Ordinary Differential Equations. Addison-Wesley Pub. Co., Reading, Mass. (1969).

26. Hurst, H. "Long-term storage capacity of reservoirs." Trans. Am. Soc. Civ. Engrs. 116, 778-808 (1951).

26-1 Iglehart, D., Whitt, W. The equivalence of functional central limit theorems for counting processes and associated partial sums." Annals of Math. Statistics 42, 1372-1378 (1971).

27. Jahnke, E., Emde, and Lösch, F. Tables of Higher Functions. McGraw-Hill Co., New York (1960).

28. Karlin, S. A First Course in Stochastic Processes. Academic Press, New York (1969).

29. Lindvaal, T. "Weak convergence on the function space $D[0,\infty)$." J. Appl. Prob. 10, 109-121 (1973).

30. Lloyd, E. "Reservoirs with serially correlated inflows." Technometrics 5, 85-93 (1963).

31. Lloyd, E. "Stochastic reservoir theory." Advances in Hydroscience Vol. 4 Ed. by Ven Te Chow. Academic Press, New York (1967).

32. McNeil, D. "A simple model for a dam in continuous time with Markovian input." Z. Wahrscheinlichkeitstheorie verw. Gebiete 21, 241-254 (1972).

33. Moran, P. The Theory of Storage. Wiley and Sons, Inc., New York (1959).

34. Moran, P. "On the range of cumulative sums." Ann. Int. Stat. Math. (Tokyo) 16, 190, 112 (1964).

35. Moyal, J. "Discontinuous Markov processes." Acta Math. 98, 221 (1957).

36. Odoom, S., and Lloyd, E. "A note on the equilibrium distribution of levels in a semi-infinite reservoir subject to Markovian inputs and unit withdrawals." J. Appl. Prob. 1, 215-222 (1965).

37. Pakes, A. "On dams with Markovian inputs." J. Appl. Prob. 10, 317-329 (1973).

38. Pinsky, M. "Differential equations with a small parameter and the central limit theorem for functions defined on a finite Markov chain." Z. Wahrscheinlichkeitstheorie verw. Gebiete 9, 101-111 (1968).

39. Resnick, S., and Durrett, R. "Functional limit-theorems for dependent variables." Ann. of Prob. Vol. 6 No. 5 (1978).

40. Scott, M. Invariant Imbedding and Its Application to Ordinary Differential Equations. Addison-Wesley Pub. Co., London (1973).

41. Troutman, B. "Limiting distributions in storage theory." Ph.D. Dissertation, Colorado State University, Fort Collins, CO., (1976).

42. Widder, D. The Laplace Transform. Princeton University Press, Princeton (1946).

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